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Antonino Salibra *

Dipartimento di Informatica, Università Ca' Foscari di Venezia, Via Torino 155, 30172 Venezia, Italy

Abstract

The variety (equational class) of *lambda abstraction algebras* was introduced to algebraize the untyped lambda calculus in the same way Boolean algebras algebraize the classical propositional calculus. The equational theory of lambda abstraction algebras is intended as an alternative to combinatory logic in this regard since it is a first-order algebraic description of lambda calculus, which allows to keep the lambda notation and hence all the functional intuitions. In this paper we show that the lattice of the subvarieties of lambda abstraction algebras is isomorphic to the lattice of lambda theories of the lambda calculus; for every variety of lambda abstraction algebras there exists exactly one lambda theory whose term algebra generates the variety. For example, the variety generated by the term algebra of the minimal lambda theory is the variety of all lambda abstraction algebras. This result is applied to obtain a generalization of the genericity lemma of finitary lambda calculus to the infinitary lambda calculus. Another result of the paper is an algebraic proof of consistency of the infinitary lambda calculus. Finally, some algebraic constructions by Krivine are generalized to lambda abstraction algebras. © 2000 Elsevier Science B.V. All rights reserved.

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0. Introduction

Although the axioms of the lambda calculus are all in the form of equations, the lambda calculus is not a true equational theory since the variable-binding properties of lambda abstraction prevent variables in lambda calculus from operating as real algebraic variables. However, there have been several attempts to reformulate the lambda calculus as a purely algebraic theory. The earliest, and best-known, algebraic models are the combinatory algebras of Curry [12] and Schönfinkel [41]. Combinatory algebras have

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* Fax: +39 041 2908419.

E-mail address: salibra@dsi.unive.it (A. Salibra).

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a simple purely equational characterization. Curry also specified (by a considerably less natural set of axioms) a purely equational subclass of combinatory algebras, the *λ-algebras* (see [3, 5.2.5]), that he viewed as algebraic models of the lambda calculus. Although lambda calculus has been the subject of research by logicians since the early 1930s, its model theory developed only much later, following the pioneering model construction made by Dana Scott. The notion of an *environment model* (the name is due to Meyer [29]) originated with Hindley and Longo [24]. They are functional domains where λ-terms can be properly interpreted. Meyer describes them as "the natural, most general formulation of what might be meant by mathematical models of the untyped lambda calculus". The main result in [29] is a completeness theorem demonstrating that every lambda theory is the theory associated with some environment model. The drawback of environment models is that they are higher-order structures. However, there exists an intrinsic characterization (up to isomorphism) of environment models as a special class of λ-algebras called *lambda models* [3, 5.2.7]. They were first axiomatized by Meyer [29] and independently by Scott [45]; the axiomatization, while elegant, is not equational. It turns out however that the variety of λ-algebras is generated by the lambda models.

In [32, 35] Pigozzi and Salibra have introduced *lambda abstraction algebras* (LAAs) which constitute a purely algebraic theory of the untyped lambda calculus alternative to Curry's highly combinatorial models. Combinatory algebras (CAs) and lambda abstraction algebras are both defined by universally quantified equations and thus form varieties in the universal algebraic sense. There are important differences, however, that result in theories of very different character. Functional application is taken as a fundamental operation in both CAs and LAAs. Lambda (i.e., functional) abstraction is also fundamental in LAAs but in CAs is defined in terms of the combinators **k** and **s**. A more important difference is connected with the role variables play in the lambda calculus as place holders. In a LAA this is also abstracted. It takes the form of a system of fundamental elements (nullary operations) of the algebra. This is a crucial feature of LAAs that has no direct analogue in CAs. One important consequence of the abstraction of variables is the abstraction of term-for-variable substitution in LAAs. Among the seven axioms characterizing LAAs, the first six constitute a recursive definition of the abstract substitution operator; they express precisely the metamathematical content of β-conversion. The last axiom is an algebraic translation of α-conversion.

The theory of lambda abstraction algebras can be regarded as axiomatizing the equations that hold between contexts of the lambda calculus, as opposed to lambda terms with free variables. We recall from Barendregt [3, Definition 14.4.1] that a context is a λ-term with some 'holes' in it. The essential feature of a context is that a free variable in a λ-term may become bound when we substitute it for a 'hole' within the context. So, Barendregt's 'holes' play the role of algebraic variables, and the contexts are the algebraic terms in the similarity type of lambda abstraction algebras. In [40] it was shown that the explicit finite equational axiomatization for the variety of LAAs provides also an explicit axiomatization of the equations between contexts valid in every lambda theory, where a lambda theory satisfies an identity between contexts if all the instances of the identity fall within the lambda theory.

In this paper a proof is given that the lattice of the subvarieties of LAAs is isomorphic to the lattice of lambda theories of the lambda calculus; for every variety of LAAs there exists exactly one lambda theory whose term algebra generates the variety. For example, the variety generated by the term algebra of the minimal lambda theory is the variety LAA of all lambda abstraction algebras, so that an identity between contexts is true in every lambda theory if and only if it is true in the minimal lambda theory. These results prove useful in the lambda calculus as a way for applying the methods of universal algebra: we can study the properties of a lambda theory by means of the variety of LAAs generated by its term algebra.

Recent work has been done by many authors on infinitary versions of lambda calculus. Berarducci defines in [5] a new model of $\lambda\beta$-calculus which is similar to the model of Böhm trees, but it does not identify all the unsolvable lambda terms. His method, that is based on an infinitary version of the lambda calculus, is also used in [6] to obtain Church–Rosser extensions of the finitary lambda calculus. Another infinitary version of lambda calculus has been independently introduced by Kenneway et al. [25]. In [40] Goldblatt and the author have shown a completeness theorem for the infinitary lambda calculus with a semantics given in terms of environment models (or lambda models). In this paper we obtain a generalization of the Genericity Lemma [3, 14.3.24] of finitary lambda calculus to the infinitary lambda calculus. We also give an algebraic proof of consistency of the infinitary lambda calculus.

In the last section of the paper we generalize some algebraic constructions by Krivine [26]. We introduce the idempotent expansions of LAAs and show that every LAA is a retract of each of its idempotent expansions. We also show that the least idempotent expansion of an LAA is an LAA.

Outline of paper. In the first section of this paper we review the basic definitions of the lambda calculus and summarize definitions and results concerning the theory of lambda abstraction algebras that will be needed in the subsequent part of the paper; in particular, we recall the formal definition of lambda abstraction algebra and the theory of abstract substitution.

The main results of the paper are presented in Section 2. We prove that the satisfiability of an identity between contexts in an LAA is equivalent to the satisfiability of a suitable identity between λ-terms. This result is the basis for the main result of the paper according to which the complete lattice of subvarieties of LAAs is isomorphic to the complete lattice of lambda theories.

In Section 3 we give an algebraic proof of consistency for the infinitary lambda calculus and generalize the Genericity Lemma from the finitary lambda calculus to the infinitary one.

The connection between the combinatory models of lambda calculus and the lambda abstraction algebras is reviewed in detail in Section 4.

Section 5 is devoted to the study of the idempotent expansions of LAAs.

1. Lambda abstraction algebras: basic notions and notation

In this section we summarize definitions and results concerning the lambda calculus and the theory of lambda abstraction algebras. Our main references will be [35, 37] for lambda abstraction algebras and Barendregt's book [3] for lambda calculus.

1.1. Lambda calculus

The untyped lambda calculus was introduced by Church [9, 10] as a foundation for logic. Although the appearance of paradoxes caused the program to fail, a consistent part of the theory turned out to be successful as a theory of "functions as rules" (formalized as terms of the lambda calculus) that stresses the computational process of going from argument to value. Every object is at the same time a function and an argument; in particular a function can be applied to itself, contrary to the usual notion of function in set theory. The two primitive notions of the lambda calculus are *application*, the operation of applying a function to an argument (expressed as juxtaposition of terms), and *lambda* (*functional*) *abstraction*, the process of forming a function from the "rule" that defines it.

The set $F_I(C)$ of ordinary terms of lambda calculus over an infinite set I of variables and a set C of constants is constructed as usual [3]: every variable $x \in I$ and every constant $c \in C$ is a λ-term; if t and s are λ-terms, then so are (st) and $\lambda x.t$ for each variable $x \in I$. We will write F_I for $F_I(\emptyset)$, the set of λ-terms without constants.

An occurrence of a variable x in a λ-term is *bound* if it lies within the scope of a lambda abstraction λx; otherwise it is *free*. A λ-term without free variables is said to be *closed*. A λ-term s is *free for* x in t if no free occurrence of x in t lies within the scope of a lambda abstraction with respect to a variable that occurs free in s. $t[s/x]$ is the result of substituting s for all free occurrences of x in t subject to the usual provisos about renaming bound variables in t to avoid capture of free variables in s. The above proviso is empty if s is free for x in t.

The axioms of the $\lambda\beta$-calculus are as follows: t and s are arbitrary λ-terms and x, y variables.
(α) $\lambda x.t = \lambda y.t[y/x]$, for any variable y that does not occur free in t;
(β) $(\lambda x.t)s = t[s/x]$, for every s free for x in t.
(β)-conversion expresses the way of calculating a function $(\lambda x.t)$ on an argument s, while (α)-conversion says that the name of bound variables does not matter. The rules for deriving equations from instances of (α) and (β) are the usual ones from equational calculus asserting that equality is a congruence for application and abstraction.

A *lambda theory* T (over $F_I(C)$) is any set of equations between λ-terms that is closed under (α) and (β) conversion and the equality rules. We will write $T \vdash t = s$ for $t = s \in T$. $\lambda\beta$ denotes the minimal lambda theory. The definition of lambda theory used here is different from the standard definition. Usually, one defines a lambda theory to be a set of equations between closed λ-terms in the language without constants (see [3, Definition 4.1.1]). Of course, every lambda theory T in our sense is determined by its restriction to closed λ-terms: for every sequence $x_1 \ldots x_n$ of variables including all the free variables of s and t,

$$T \vdash t = s \text{ if, and only if, } T \vdash \lambda x_1 \ldots x_n.t = \lambda x_1 \ldots x_n.s.$$

1.2. Lambda abstraction algebras

Let I be a nonempty set. The similarity type of *lambda abstraction algebras of dimension I* is $\langle \cdot, \langle \lambda x : x \in I \rangle, \langle x : x \in I \rangle \rangle$, where "$\cdot$" is a binary operation symbol formalizing application, λx is a unary operation symbol for every $x \in I$, and x is a constant symbol (i.e., nullary operation symbol) for every $x \in I$. Note that λx is to be viewed as an indivisible symbol. The elements of I are the variables of lambda calculus although in their algebraic transformation they no longer play the role of variables in the usual sense. In the remaining part of the paper we will refer to them as *λ-variables*. The actual variables of the lambda abstraction theory will be referred to as *context variables* and denoted by the greek letters ξ, v, and μ, possibly with subscripts. The terms of the language of lambda abstraction theory are called *λ-contexts*. They are constructed in the usual way: every λ-variable x and context variable ξ is a λ-context; if t and s are λ-contexts, then so are $t \cdot s$ and $\lambda x(t)$. Because of their similarity to the terms of the lambda calculus we use the standard notational conventions of the latter. The application operation symbol "$\cdot$" is normally omitted, and the application of t and s is written as juxtaposition ts. When parentheses are omitted, association to the left is assumed. The left parenthesis delimiting the scope of a lambda abstraction is replaced with a period and the right parenthesis is omitted. For example, $\lambda x(ts)$ is written $\lambda x.ts$. Successive λ-abstractions $\lambda x \lambda y \lambda z \cdots$ are written $\lambda xyz \cdots$.

We say that a λ-context t is *over* $\bar{x}$ if $\bar{x} = x_1 \ldots x_k$ is a finite sequence of λ-variables which contains all the λ-variables occurring in t either as constants x_i or as λ-abstractions λx_i. An occurrence of a λ-variable x in a λ-context is *bound* if it falls within the scope of the operation symbol λx; otherwise it is *free*. The *free λ-variables* of a λ-context are the λ-variables that have at least one free occurrence. A λ-context without free λ-variables is said to be *closed*. Note that λ-contexts without any context variables coincide with ordinary terms of the lambda calculus without constants.

A word of caution for those readers familiar with the lambda calculus. When dealing with models of the lambda calculus one often allows terms that contain constant symbols representing the elements of the models. These constants should not be confused with context variables; they play a much different role. Our notion of a λ-context coincides with the notion of *context* defined in [3, Definition 14.4.1]; our context variables correspond to Barendregt's notion of a 'hole'. The main difference between Barendregt's notation and our's is that 'holes' are denoted here by Greek letters $\xi, \mu, \ldots$, while in Barendregt's book by $[\ ], [\ ]_1, \ldots$. The essential feature of a λ-context is that a free λ-variable in a λ-term may become bound when we substitute it for a 'hole' within the context. For example, if $C(\xi) = \lambda x.x(\lambda y.\xi)$ is a λ-context, in Barendregt's notation: $C([\ ]) = \lambda x.x(\lambda y.[\ ])$, and $t = xy$ is a λ-term, then $C(t) = \lambda x.x(\lambda y.xy)$.

Let T be a lambda theory over the language $F_I(C)$ and let $\mathbf{F}_I(C)$ be the absolutely free algebra in the similarity type of lambda abstraction algebras (of dimension I) over the set C of generators, i.e.,

$$\mathbf{F}_I(C) := \langle F_I(C), \cdot^{\mathbf{F}_I(C)}, \{\lambda x^{\mathbf{F}_I(C)} : x \in I\}, \{x^{\mathbf{F}_I(C)} : x \in I\}\rangle, \tag{1.1}$$

where for $s, t \in F_I(C)$

$$s \cdot^{\mathbf{F}_I(C)} t = (st); \quad \lambda x^{\mathbf{F}_I(C)}(t) = \lambda x.t; \quad x^{\mathbf{F}_I(C)} = x.$$

We will write $\mathbf{F}_I$ for $\mathbf{F}_I(\emptyset)$, the absolutely free algebra over an empty set of generators. The lambda theory T is a congruence on $\mathbf{F}_I(C)$. We denote by $\mathbf{F}_I^T$ the quotient of $\mathbf{F}_I(C)$ by T and call it the *term algebra* of the lambda theory T. We say that T satisfies an identity between λ-contexts

$$t(\xi_1, \ldots, \xi_n) = u(\xi_1, \ldots, \xi_n)$$

if the term algebra $\mathbf{F}_I^T$ of T satisfies it, i.e., if all the instances of the above identity, obtained by substituting λ-terms for context variables in it, fall within the lambda theory:

$$T \vdash t(t_1, \ldots, t_n) = u(t_1, \ldots, t_n) \quad \text{for all } \lambda\text{-terms } t_1, \ldots, t_n \in F_I(C).$$

For example, every lambda theory satisfies the identity $(\lambda x.x)\xi = \xi$ because $\lambda\beta \vdash (\lambda x.x)t = t$ for every λ-term t.

Lambda abstraction algebras are meant to axiomatize those identities between λ-contexts that are valid for the lambda calculus. We now give the formal definition of a lambda abstraction algebra. Readers unfamiliar with the notation of the lambda calculus may want to go directly to the reformulation of the axioms in terms of the substitution operations that is given later.

Definition 1. By a *lambda abstraction algebra of dimension I* we mean an algebraic structure of the form

$$\mathbf{A} := \langle A, \cdot^{\mathbf{A}}, \{\lambda x^{\mathbf{A}} : x \in I\}, \{x^{\mathbf{A}} : x \in I\}\rangle$$

satisfying the following identities for all $x, y, z \in I$ and all $\xi, \mu, \nu \in A$ (we simplify the notation by suppressing the $\mathbf{A}$-superscript):

(β_1) $(\lambda x.x)\xi = \xi$;
(β_2) $(\lambda x.y)\xi = y,\ x \neq y$;
(β_3) $(\lambda x.\xi)x = \xi$;
(β_4) $(\lambda xx.\xi)\mu = \lambda x.\xi$;
(β_5) $(\lambda x.\xi\mu)\nu = (\lambda x.\xi)\nu((\lambda x.\mu)\nu)$;
(β_6) $(\lambda xy.\mu)((\lambda y.\xi)z) = \lambda y.(\lambda x.\mu)((\lambda y.\xi)z),\ x \neq y,\ z \neq y$;
(α) $\lambda x.(\lambda y.\xi)z = \lambda y.(\lambda x.(\lambda y.\xi)z)y,\ z \neq y$.

I is called the *dimension set* of $\mathbf{A}$. $\cdot^{\mathbf{A}}$ is called *application* and $\lambda x^{\mathbf{A}}$ is called *λ-abstraction* with respect to x.

The class of lambda abstraction algebras of dimension I is denoted by LAA_I and the class of all lambda abstraction algebras of any dimension by LAA. We also use LAA_I as shorthand for the phrase "lambda abstraction algebra of dimension I", and similar for LAA. An LAA_I is *infinite dimensional* if I is infinite.

LAA_I is a variety for every dimension set I, and therefore is closed under the formation of subalgebras, homomorphic (in particular isomorphic) images, and Cartesian products. In symbols $\mathbb{S}\ \mathsf{LAA}_I = \mathbb{H}\ \mathsf{LAA}_I = \mathbb{I}\ \mathsf{LAA}_I = \mathbb{P}\ \mathsf{LAA}_I = \mathsf{LAA}_I$.

The following result can be easily verified.

Proposition 2. *Let T be a lambda theory over the language $F_I(C)$. Then the term algebra $\mathbf{F}_I^T$ of T is an* LAA_I.

We note here one very useful immediate consequence of the axioms: in any LAA_I $\mathbf{A}$ the functions λx are always one–one, i.e., for all $x \in I$,

$$\lambda x.a = \lambda x.b \quad \text{iff } a = b \quad \text{for all } a, b \in A.$$

For if $\lambda x.a = \lambda x.b$, then by β_3, $a = (\lambda x.a)x = (\lambda x.b)x = b$.

An LAA with only one element is said to be *trivial*. It is interesting that any non-trivial LAA_I $\mathbf{A}$ of positive dimension is infinite, since the one–one map λx is not onto. To see this, assume by way of contradiction that x is in the range of λx; then $x = \lambda x.b$ for some element $b \in A$. Since $\mathbf{A}$ is nontrivial, there exists an element $a \in A$ such that $a \neq x$. Then a contradiction results from β_1 and β_4: $a = (\lambda x.x)a = (\lambda xx.b)a = \lambda x.b = x$.

The equations defining LAA_Is express algebraically various instances of (α) and (β) conversion. When transformed into the equational language of lambda abstraction theory, (β)-conversion becomes the definition of abstract substitution: $S_b^x(a) = (\lambda x.a)b$, which can be interpreted as "a with b substituted for the free occurrences of x".

It is obvious that the axioms for lambda abstraction algebras can be reformulated in the following way:

(β_1) $S_\xi^x(x) = \xi$;

(β_2) $S_\xi^x(y) = y, \ y \neq x$;

(β_3) $S_x^x(\xi) = \xi$;

(β_4) $S_\mu^x(\lambda x.\xi) = \lambda x.\xi$;

(β_5) $S_\nu^x(\xi\mu) = S_\nu^x(\xi)S_\nu^x(\mu)$;

(β_6) $S^x_{S_z^y(\xi)}(\lambda y.\mu) = \lambda y.S^x_{S_z^y(\xi)}(\mu), \ x \neq y, \ z \neq y$;

(α) $\lambda x.S_z^y(\xi) = \lambda y.S_y^x(S_z^y(\xi)), \ z \neq y$.

A λ-term t does not admit free occurrences of a λ-variable x if t is the result of the process of substituting an arbitrary λ-variable $z \neq x$ for the free occurrences of x in t. This process is abstracted in this way.

Definition 3. Let $\mathbf{A}$ be an LAA_I. Let $a \in A$ and $x \in I$. a is said to be *algebraically dependent on x (over* $\mathbf{A}$) if $(\lambda x.a)z \neq a$ for some $z \in I$; otherwise a is *algebraically* *independent of* x (*over* $\mathbf{A}$). The set of all $x\in I$ such that a is algebraically dependent on x over $\mathbf{A}$ is called the *dimension set* of a and is denoted by $\boldsymbol{\Delta}^{\mathbf{A}}a$; thus

$$\boldsymbol{\Delta}^{\mathbf{A}}a=\{x\in I:(\lambda x.a)z\neq a \text{ for some } z\in I\}.$$

a is *finite* (*infinite*) *dimensional* if $\boldsymbol{\Delta}a$ is finite (infinite). An element a is called *zero dimensional if* $\boldsymbol{\Delta}a=\emptyset$. We denote the set of zero-dimensional elements by $\mathrm{Zd}\,\mathbf{A}$.

In the following three propositions we give some basic properties of substitution and dimension set. The proofs of Propositions 4–6 can be found in [35].

Proposition 4. *Let I be a nonempty set with $|I|\geqslant 2$. Let $\mathbf{A}\in \mathsf{LAA}_I$, $a\in A$, and $x\in I$. The following are equivalent*:
(i) $S^x_z(a)=a$ *for some* $z\in I\backslash\{x\}$;
(ii) $S^x_z(a)=a$ *for all* $z\in I$, *i.e.*, $x\notin\boldsymbol{\Delta}a$;
(iii) $S^x_b(a)=a$ *for all* $b\in A$.

It is immediate from β_2, β_4 and β_5 that S^x_z is idempotent, and hence the set of all elements of $\mathbf{A}$ algebraically independent of x is equal to $\{S^x_z(b):b\in A\}$ $(z\neq x)$. Then identities (β_6) and (α) can be reformulated as follows.

For all $\xi\in A$ independent of y:
(β_6) $S^x_\xi(\lambda y.\mu)=\lambda y.S^x_\xi(\mu)$, $x\neq y$;
(α) $\lambda x.\xi=\lambda y.S^x_y(\xi)$.

Proposition 5. *Let $\mathbf{A}\in\mathsf{LAA}_I$, $a,b\in A$, and $x\in I$.*
(i) $\boldsymbol{\Delta}(ab)\subseteq\boldsymbol{\Delta}a\cup\boldsymbol{\Delta}b$.
(ii) $\boldsymbol{\Delta}(\lambda x.a)=\boldsymbol{\Delta}a\backslash\{x\}$.
(iii) $\boldsymbol{\Delta}(S^x_b(a))\subseteq(\boldsymbol{\Delta}a\backslash\{x\})\cup\boldsymbol{\Delta}b$.
(iv) $\boldsymbol{\Delta}x\subseteq\{x\}$, *with equality holding if* $\mathbf{A}$ *is nontrivial.*

If t is a λ-term without constants and $\mathbf{A}$ is an LAA_I, then $t^{\mathbf{A}}$ will denote the value of t in $\mathbf{A}$ when each λ-variable x occurring in t is interpreted as $x^{\mathbf{A}}$. By Proposition 5 the dimension set of $t^{\mathbf{A}}$ is a subset of the set of free λ-variables of t.

For any set B, B^* denotes the set of all finite strings of elements of B with repetitions, while $B^\star\subseteq B^*$ denotes the subset of all finite strings without repetitions.

Proposition 6. *Let $\mathbf{A}$ be an LAA_I, $\mathbf{x}=x_1\cdots x_n\in I^\star$, and $\mathbf{b}=b_1\cdots b_n\in A^*$. If b_i is independent of $x_1,\dots,x_{i-1}$ for $i=2,\dots,n$, in particular, if each b_i is independent of all the x_j, then*

$$S^{x_1}_{b_1}(S^{x_2}_{b_2}(\dots S^{x_n}_{b_n}(a)\dots))=(\lambda x_1\cdots x_n.a)b_1\cdots b_n \quad \textit{for all } a\in A.$$

Locally finite LAAs. There is a strong connection between the lambda theories and the subclass of LAAs whose elements are finite dimensional.

Definition 7. A lambda abstraction algebra **A** is *locally finite* if it is of infinite dimension (i.e., I is infinite) and every $a \in A$ is of finite dimension (i.e., $|\Delta a| < \omega$).

The class of locally finite LAA_Is is denoted by LFA_I, which is also used as shorthand for the phrase "locally finite lambda abstraction algebra of dimension I".

For every infinite I the term algebra $\mathbf{F}_I^T$ of a lambda theory T is locally finite. This is a direct consequence of the trivial fact that every λ-term is a finite string of symbols and hence contains only finitely many λ-variables.

Recall that $\mathbf{F}_I(C)$ is the absolutely free algebra in the similarity type of LAA_Is over a set C of generators (see (1.1) above). It is not an LAA_I. The following result, characterizing those congruences on $\mathbf{F}_I(C)$ that are lambda theories, will be repeatedly used in the sequel.

Lemma 8. *Let I be an infinite set. A congruence θ on $\mathbf{F}_I(C)$ is a lambda theory over the language $F_I(C)$ if, and only if, the following two conditions are satisfied*:
(i) *The quotient algebra $\mathbf{F}_I(C)/\theta$ is an LAA_I*;
(ii) *$(\lambda x.c)y\theta c$ for all $c \in C$ and all $x, y \in I$, i.e., the equivalence class c/θ of every element $c \in C$ is a zero-dimensional element of $\mathbf{F}_I(C)/\theta$.*

The following proposition is the algebraic analog of [26, Propositions 1 and 3, Chapter VII].

Proposition 9 (Pigozzi and Salibra [37, Proposition 2.4]). *Let I be an infinite set. An algebra **A** in the similarity type of lambda abstraction algebras of dimension I is* (*isomorphic to*) *the term algebra of a lambda theory if, and only if, it is an LFA_I.*

Proof. We outline the nontrivial part of the proof. Let **A** be an LFA_I and let $\mathrm{Zd}\,\mathbf{A}$ be the set of zero-dimensional elements of **A**. Consider the unique homomorphism h from the absolutely free algebra $\mathbf{F}_I(\mathrm{Zd}\,\mathbf{A})$ into **A** that is the identity on $\mathrm{Zd}\,\mathbf{A}$. The map h is onto. Then the LFA_I **A** is isomorphic to the quotient algebra $\mathbf{F}_I(\mathrm{Zd}\,\mathbf{A})/\theta$, where the congruence θ is the relation-kernel of h (i.e., $t\ \theta\ u$ iff $h(t) = h(u)$). The conclusion of the proposition is a consequence of Lemma 8. $\square$

The last two propositions of this section have not been explicitly stated elsewhere.

Recall from [8, Definition 10.9] that the LAA_I-free algebra over an empty set of generators is the quotient of the absolutely free algebra $\mathbf{F}_I$ of λ-terms by the smallest congruence θ making $\mathbf{F}_I/\theta$ an LAA_I.

Proposition 10. *Let I be an infinite set. The term algebra $\mathbf{F}_I^{\lambda\beta}$ of the minimal lambda theory $\lambda\beta$ is the LAA_I-free algebra over an empty set of generators.*

Proof. Let θ be the smallest congruence making $\mathbf{F}_I/\theta$ an LAA_I. By Propostion 2 we have that $\theta \subseteq T$ for every lambda theory T over F_I. To obtain the conclusion of the proposition, namely $\lambda\beta = \theta$, it suffices to show that θ is a lambda theory, i.e., that it is closed under (α) and (β)-conversion. But this is a consequence of Lemma 8 and of the hypothesis $\mathbf{F}_I/\theta \in \mathsf{LAA}_I$. □

Proposition 11. *Let I be an infinite set. For all λ-terms* $t, u \in F_I$, $\mathsf{LAA}_I \models t = u$ *iff* $\lambda\beta \vdash t = u$.

Proof. By a well-known result of elementary universal algebra an identity between λ-terms (i.e., λ-contexts without context variables) holds in the variety LAA_I if, and only if, it holds in the LAA_I-free algebra over an empty set of generators. The conclusion is a consequence of Proposition 10. □

2. The lattice of subvarieties of LAA_Is is isomorphic to the lattice of lambda theories

In this section we show that the complete lattice of subvarieties of LAA_Is is isomorphic to the complete lattice of lambda theories over the language F_I. For every variety $\mathscr{V}$ of LAA_Is there exists exactly one lambda theory T over F_I such that $\mathscr{V}$ is the variety generated by the term algebra of T. The variety generated by the term algebra of the minimal lambda theory $\lambda\beta$ is the variety LAA_I of lambda abstraction algebras of dimension I. The main result is applied in Section 3 to obtain a generalization of the genericity lemma of finitary lambda calculus to the infinitary lambda calculus.

Let $t(\xi_1, \dots, \xi_n)$ be a λ-context over $\bar{x}$ (i.e., $\bar{x} = x_1 \dots x_k$ is the finite sequence of λ-variables which contains all the λ-variables occurring in t either as constants x_i or as λ-abstractions λx_i). Let $\bar{y} = y_1 \dots y_n$ be an n-tuple of λ-variables such that $\bar{y} \cap \bar{x} = \emptyset$. Define

$$t(y_1 x_1 \dots x_k, \dots, y_n x_1 \dots x_k)$$

as the λ-term obtained from the λ-context t by substituting the λ-term $y_i x_1 \dots x_k$ for all the occurrences of the context variable ξ_i in t ($i = 1, \dots, n$). (Recall that $y_i x_1 \dots x_k$ means $(\dots((y_i x_1)x_2)\dots)x_k)$.)

If $\bar{y} = y_1 \cdots y_n$ is a sequence of λ-variables and $\bar{\xi} = \xi_1 \dots \xi_n$ is a sequence of context variables, we will write $\lambda\bar{y}$ for $\lambda y_1 \cdots y_n$; $t(\bar{\xi})$ for $t(\xi_1, \dots, \xi_n)$; and $t(y_1\bar{x}, \dots, y_n\bar{x})$ for $t(y_1 x_1 \dots x_k, \dots, y_n x_1 \dots x_k)$. We always assume that $\bar{\xi}$ and $\bar{y}$ have the same length.

Lemma 12. *Let* $\mathbf{A}$ *be an* LAA_I. *Let* $t(\bar{\xi})$ *be a λ-context over* $\bar{x} = x_1 \cdots x_k$ *and let* $\bar{y} = y_1 \cdots y_n$ *such that* $\bar{y} \cap \bar{x} = \emptyset$. *For all* $a \in A$ *with* $\bar{x} \notin \Delta a$ *and all* $b_1, \dots, b_n \in A$ *we have*

$$S_a^{y_i}(t^{\mathbf{A}}(b_1, \dots, b_n)) = t^{\mathbf{A}}(S_a^{y_i}(b_1), \dots, S_a^{y_i}(b_n)).$$

Proof. The proof is by induction over the complexity of t using β_1, β_2, β_5, β_6 and $\bar{x} \notin \Delta a$. We provide the only two nontrivial cases.

$(t \equiv t_1 t_2)$:

$$\begin{aligned} S_a^{y_i}[t^{\mathbf{A}}(b_1,\ldots,b_n)] &= S_a^{y_i}[t_1^{\mathbf{A}}(b_1,\ldots,b_n)t_2^{\mathbf{A}}(b_1,\ldots,b_n)] \\ &= (S_a^{y_i}[t_1^{\mathbf{A}}(b_1,\ldots,b_n)])(S_a^{y_i}[t_2^{\mathbf{A}}(b_1,\ldots,b_n)]) \quad [(\beta_5)] \\ &= (t_1^{\mathbf{A}}(S_a^{y_i}(b_1),\ldots,S_a^{y_i}(b_n)))(t_2^{\mathbf{A}}(S_a^{y_i}(b_1),\ldots,S_a^{y_i}(b_n))) \\ &\qquad \text{[Induction]} \\ &= t^{\mathbf{A}}(S_a^{y_i}(b_1),\ldots,S_a^{y_i}(b_n)). \end{aligned}$$

$(t \equiv \lambda x.t_1)$:

$$\begin{aligned} S_a^{y_i}[t^{\mathbf{A}}(b_1,\ldots,b_n)] &= S_a^{y_i}[\lambda x^{\mathbf{A}}.t_1^{\mathbf{A}}(b_1,\ldots,b_n)] \\ &= \lambda x^{\mathbf{A}}.S_a^{y_i}[t_1^{\mathbf{A}}(b_1,\ldots,b_n)]) \quad [(\beta_6,\ \bar{x} \notin \Delta a)] \\ &= \lambda x^{\mathbf{A}}.t_1^{\mathbf{A}}(S_a^{y_i}(b_1),\ldots,S_a^{y_i}(b_n)) \quad \text{[induction]} \\ &= t^{\mathbf{A}}(S_a^{y_i}(b_1),\ldots,S_a^{y_i}(b_n)). \qquad \square \end{aligned}$$

The author is indebted to an anonymous referee for providing the following short syntactic proof of Theorem 13. A *self-contained* proof, such as the one given below, is interesting because it implies one of the main results in [40] (see Theorem 29 below) with a proof which is considerably easier than the one contained in [40].

Theorem 13. *Let* $\mathbf{A}$ *be an infinite dimensional* LAA_I. *Let* $t(\bar{\xi})$, $u(\bar{\xi})$ *be* λ*-contexts over* $\bar{x} = x_1 \cdots x_k$ *and let* $\bar{y} = y_1 \cdots y_n$ *such that* $\bar{y} \cap \bar{x} = \emptyset$. *Then,*

$$\mathbf{A} \models t(\bar{\xi}) = u(\bar{\xi}) \text{ if and only if } \mathbf{A} \models t(y_1\bar{x},\ldots,y_n\bar{x}) = u(y_1\bar{x},\ldots,y_n\bar{x}).$$

Proof. Assume for a moment that the LAA_I $\mathbf{A}$ satisfies the following identity:

$$t(\bar{\xi}) = (\lambda\bar{y}.t(y_1\bar{x},\ldots,y_n\bar{x}))(\lambda\bar{x}.\xi_1)\ldots(\lambda\bar{x}.\xi_n). \tag{2.1}$$

The equational calculus is closed under contexts. If

$$\mathbf{A} \models t(y_1\bar{x},\ldots,y_n\bar{x}) = u(y_1\bar{x},\ldots,y_n\bar{x}),$$

then

$$\mathbf{A} \models (\lambda\bar{y}.t(y_1\bar{x},\ldots,y_n\bar{x}))(\lambda\bar{x}.\xi_1)\ldots(\lambda\bar{x}.\xi_n) = (\lambda\bar{y}.u(y_1\bar{x},\ldots,y_n\bar{x}))(\lambda\bar{x}.\xi_1)\ldots(\lambda\bar{x}.\xi_n).$$

The conclusion of the theorem is now an immediate consequence of (2.1). So, we have to show that every LAA_I satisfies (2.1).

First, notice that for $n = 0$, there is nothing to show. Therefore, assume $n > 0$. Let z be a λ-variable such that $z \notin \bar{y}$. We utilize the following abbreviations:

$$\begin{aligned} T &= t(y_1\bar{x},\ldots,y_n\bar{x}), \\ V_i &= y_1(\lambda\bar{y}\,\bar{x}.\xi_i), \\ C &= \lambda z.z\bar{y}. \end{aligned}$$

Notice that $\bar{x} \notin \mathbf{\Delta}(V_i)$, $y_2 \dots y_n \notin \mathbf{\Delta}(V_i)$, for $i = 1, \dots, n$, and $\bar{x} \notin \mathbf{\Delta}(C)$. Also notice that for $i = 1, \dots, n$,

$$S_C^{y_1} V_i = \lambda \bar{x}.\xi_i, \tag{2.2}$$

because

$$\begin{aligned} S_C^{y_1} V_i &= C(\lambda \bar{y}\bar{x}.\xi_i) \quad [(\beta_5), (\beta_1), (\beta_4)] \\ &= (S^z_{\lambda \bar{y}\bar{x}.\xi_i} z \bar{y}) \quad [\text{def } C] \\ &= (\lambda \bar{y}\bar{x}.\xi_i)\bar{y} \quad [(\beta_5), (\beta_1), (\beta_2)] \\ &= \lambda \bar{x}.\xi_i \quad [(\beta_3)] \end{aligned}$$

Now consider the λ-context

$$B = S_C^{y_1}[(\lambda \bar{y}.T)V_1 \dots V_n].$$

One has

$$\begin{aligned} B &= (\lambda \bar{y}.T)(S_C^{y_1} V_1) \dots (S_C^{y_1} V_n) \quad [(\beta_5), (\beta_4)] \\ &= (\lambda \bar{y}.T)(\lambda \bar{x}.\xi_1) \dots (\lambda \bar{x}.\xi_n)) \quad [(2.2)] \end{aligned}$$

which is the right-hand side of (2.1). On the other hand,

$$\begin{aligned} B &= S_C^{y_1}[(\lambda y_1 \dots y_n.T)V_1 \dots V_n] \quad [\text{def } B] \\ &= S_C^{y_1}[(S_{V_1}^{y_1}(\lambda y_2 \dots y_n.T))V_2 \dots V_n] \quad [\text{def } S_{V_1}^{y_1}] \\ &= S_C^{y_1}[(\lambda y_2 \dots y_n.S_{V_1}^{y_1} T)V_2 \dots V_n] \quad [(\beta_6), y_2 \cdots y_n \notin \mathbf{\Delta}(V_1)] \\ &= S_C^{y_1}[S_{V_n}^{y_n} \dots S_{V_1}^{y_1}(T)] \quad [(\beta_6), y_2 \dots y_n \notin \mathbf{\Delta}(V_2, \dots, V_n)] \\ &= S_C^{y_1}[S_{V_n}^{y_n} \dots S_{V_1}^{y_1}(t(y_1\bar{x}, \dots, y_n\bar{x}))] \quad [\text{def } T] \\ &= S_C^{y_1}[t(V_1\bar{x}, \dots, V_n\bar{x})] \quad [\text{Lemma 12}] \\ &= t(S_C^{y_1}(V_1\bar{x}), \dots, S_C^{y_1}(V_n\bar{x})) \quad [\text{Lemma 12}] \\ &= t((\lambda \bar{x}.\xi_1)\bar{x}, \dots, (\lambda \bar{x}.\xi_n)\bar{x}) \quad [(2.2), (\beta_5), (\beta_2)] \\ &= t(\xi_1, \dots, \xi_n) \quad [(\beta_3)]. \end{aligned}$$

This finishes the proof. □

The proof did not use the fact that I was infinite; in fact, for each pair $t(\bar{\xi})$, $u(\bar{\xi})$ of λ-contexts over $\bar{x}$, it used no λ-variables other than $\bar{x}$, $\bar{y}$ and z. Note that it is possible that $k = 0$. If $k > 0$, one may even take $z \in \bar{x}$.

Let $\mathscr{V}$ be an arbitrary variety of algebras and $\mathbf{A} \in \mathscr{V}$. Then $\mathbf{A}$ is said to be *generic* (*in* $\mathscr{V}$) if an identity holds in $\mathbf{A}$ iff it holds in $\mathscr{V}$; equivalently, $\mathbf{A}$ is generic iff it generates $\mathscr{V}$ as a variety (see [19, p. 383]).

Recall from Section 1 that, if T is a lambda theory we denote by $\mathbf{F}_I^T$ the term algebra of the lambda theory T. So, $\mathbf{F}_I^{\lambda\beta}$ is the term algebra of the minimal lambda theory $\lambda\beta$.

Theorem 14. *For any infinite set I, the variety generated by the term algebra $\mathbf{F}_I^{\lambda\beta}$ of the minimal lambda theory $\lambda\beta$ is the variety of LAA_Is, in symbols,*

$$\mathsf{LAA}_I = \mathbb{HSP}(\mathbf{F}_I^{\lambda\beta}).$$

Every LAA_I admitting a subalgebra isomorphic to $\mathbf{F}_I^{\lambda\beta}$ is generic in the variety LAA_I.

Proof. We need to show that if an equation between λ-contexts holds in $\mathbf{F}_I^{\lambda\beta}$, then it holds in all LAA_I's. Thus, let $t(\bar{\xi}) = u(\bar{\xi})$ be an equation valid for $\mathbf{F}_I^{\lambda\beta}$, where $t(\bar{\xi})$ and $u(\bar{\xi})$ are λ-contexts over $\bar{x}$. By Theorem 13 this is equivalent to

$$\mathbf{F}_I^{\lambda\beta} \models t(y_1\bar{x}, \dots, y_n\bar{x}) = u(y_1\bar{x}, \dots, y_n\bar{x}), \tag{2.3}$$

where $\bar{y} = y_1 \dots y_n$ is any sequence of λ-variables such that $\bar{y} \cap \bar{x} = \emptyset$. Let $\mathbf{A}$ be an arbitrary LAA_I. Since by Proposition 10 the term algebra $\mathbf{F}_I^{\lambda\beta}$ is the LAA_I-free algebra over an empty set of generators, then it is an initial object in the category of LAA_Is [8, Theorem 10.10]; hence there exists a unique homomorphism h from $\mathbf{F}_I^{\lambda\beta}$ into $\mathbf{A}$. The image of h is a subalgebra $\mathbf{B}$ of $\mathbf{A}$. By (2.3) and by well-known properties of homomorphisms we obtain

$$\mathbf{B} \models t(y_1\bar{x}, \dots, y_n\bar{x}) = u(y_1\bar{x}, \dots, y_n\bar{x}).$$

Since this equation contains no context variables and $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, it follows that

$$\mathbf{A} \models t(y_1\bar{x}, \dots, y_n\bar{x}) = u(y_1\bar{x}, \dots, y_n\bar{x})$$

and thus, by Theorem 13,

$$\mathbf{A} \models t(\bar{\xi}) = u(\bar{\xi}).$$

This finishes the first part of the proof.

Let now $\mathbf{A}$ be an LAA_I having $\mathbf{F}_I^{\lambda\beta}$ as a subalgebra. To prove the last part of the theorem it is sufficient to check that $\mathbf{A}$ and $\mathbf{F}_I^{\lambda\beta}$ satisfy exactly the same equations.

$$\begin{aligned} \mathbf{A} \models t(\bar{\xi}) = u(\bar{\xi}) \text{ iff } & \mathbf{A} \models t(y_1\bar{x}, \dots, y_n\bar{x}) = u(y_1\bar{x}, \dots, y_n\bar{x}) \quad \text{[Theorem 13]} \\ \text{iff } & \mathbf{F}_I^{\lambda\beta} \models t(y_1\bar{x}, \dots, y_n\bar{x}) = u(y_1\bar{x}, \dots, y_n\bar{x}) \\ \text{iff } & \mathbf{F}_I^{\lambda\beta} \models t(\bar{\xi}) = u(\bar{\xi}) \quad \text{[Theorem 13].} \end{aligned}$$

The second equivalence holds because $\mathbf{F}_I^{\lambda\beta}$ is a subalgebra of $\mathbf{A}$ and the identity $t(y_1\bar{x}, \dots, y_n\bar{x}) = u(y_1\bar{x}, \dots, y_n\bar{x})$ contains no context variables. □

The above theorem implies one of the main results in [40].

Theorem 15 (Salibra and Goldblatt [40]). *For any infinite set I,*

$$\mathsf{LAA}_I = \mathbb{HSP}\ \mathsf{LFA}_I,$$

i.e. the variety LAA_I *is generated by its locally finite members.*

Proof. By Theorem 14 and by $\{\mathbf{F}_I^{\lambda\beta}\} \subseteq \mathsf{LFA}_I \subseteq \mathsf{LAA}_I$. □

Since every LFA_I is isomorphic to a term algebra (Proposition 9), then from Theorems 15 and 14 it follows that the class of identities between λ-contexts true in every lambda theory is equal to the class of identities true in $\lambda\beta$; it is axiomatized by the finitely many schemes of identities characterizing the variety LAA_I.

It is obvious that there exists a one-to-one correspondence between the set of lambda theories over the set F_I of λ-terms (without constants) and the set of congruences over the term algebra of the minimal lambda theory $\lambda\beta$. So, the set of lambda theories over F_I constitutes a complete lattice.

We now characterize the lattice of subvarieties of the variety LAA_I.

Theorem 16. *Let* $\mathscr{V}$ *be a subvariety of the variety* LAA_I. *Then there exists exactly one lambda theory T over* F_I *such that the term algebra* $\mathbf{F}_I^T$ *is generic in* $\mathscr{V}$:

$$\mathscr{V} = \mathbb{HSP}(\mathbf{F}_I^T).$$

Every LAA_I *admitting a subalgebra isomorphic to* $\mathbf{F}_I^T$ *is generic in the variety* $\mathscr{V}$.

Proof. Since $\mathscr{V}$ is a subvariety of LAA_I, it follows from a well-known result of elementary universal algebra that the $\mathscr{V}$-free algebra with an empty set of generators is a homomorphic image of $\mathbf{F}_I^{\lambda\beta}$, the LAA_I-free algebra with an empty set of generators. Hence, there exists a lambda theory T such that $\mathbf{F}_I^T$ is the $\mathscr{V}$-free algebra with an empty set of generators. The conclusion of the theorem can be now obtained as in Theorem 14. □

The following theorem is now immediate.

Theorem 17. *There is a complete lattice isomorphism between the lattice of subvarieties of* LAA_I *and the lattice of lambda theories over* F_I (*or the lattice of congruences over the term algebra of the minimal lambda theory* $\lambda\beta$).

The variety generated by the term algebra of a lambda theory T will be denoted by LAA_I^T.

A variety is called equationally complete if it contains no proper, nontrivial subvarieties, in other words, it is a minimal (nontrivial) variety.

We recall from [3, Definition 4.1.22] that a lambda theory T is called Hilbert–Post (HP)-complete if, for every equation $t = u$ between λ-terms in the language of T, we have $T \vdash t = u$ or $T \cup \{t = u\}$ is inconsistent.

Corollary 18. *The equationally complete varieties of* LAA_I*s are exactly the varieties generated by the term algebras of the HP-complete lambda theories over the language* F_I.

A lambda theory is sensible [3, Definition 16.1.1] if it contains all equations of the form $t = u$, where t and u are closed, unsolvable λ-terms. A closed λ-term u is unsolvable if there exist no λ-terms $t_1, \dots, t_n$ such that $ut_1 \cdots t_n = \lambda x.x$. It turns out that sensible lambda theories admit a unique HP-complete extension H^* [3, Theorem 16.2.6]. The term algebra $\mathbf{F}_I^{H^*}$ associated with H^* is simple (i.e., it does not admit nontrivial congruences) and has no proper subalgebras. By Corollary 18 we have that the variety $\mathsf{LAA}_I^{H^*}$ generated by $\mathbf{F}_I^{H^*}$ is equationally complete.

3. The infinitary lambda calculus

Many authors have recently worked on infinitary versions of lambda calculus. Berarducci defines in [5] a new model of lambda calculus which is similar to the model of Böhm trees, but it does not identify all the unsolvable lambda terms. His method, which is based on an infinitary version of lambda calculus, is also used in [6] to obtain Church–Rosser extensions of the finitary lambda calculus. Another infinitary version of the lambda calculus has been independently introduced by Kenneway et al. [25]. In [40], a completeness theorem was shown for the infinitary lambda calculus with a semantics given in terms of environment models.

The main result of this section is an algebraic proof of consistency of the infinitary lambda calculus. We also obtain a generalization of the genericity lemma of finitary lambda calculus to the infinitary lambda calculus.

Let I be an infinite set of λ-variables and $\perp$ a new symbol. An *infinitary λ-term over I* (see [5]) is defined as a finite or infinite rooted tree such that each leaf is either labeled by a λ-variable in I or by the constant $\perp$, and the inner nodes are either binary 'application nodes', or unary 'abstraction nodes', in which case they have a label of the form λx where $x \in I$ is a λ-variable. The set of infinitary λ-terms is denoted by F_I^∞. The infinitary λ-terms include as special cases the finitary ones. The notion of free and bound occurrence of a λ-variable is easily extended to infinitary λ-terms. We write infinitary λ-terms in their linear form. Unless otherwise stated 'λ-term' means 'finitary λ-term'.

Infinitary λ-terms arise as 'limits' of infinite sequences of β-reductions. For example, let $\omega_3 = \lambda x.xxx$ and $\Omega_3 = \omega_3\omega_3$. If we start with Ω_3 we can generate the infinite sequence of β-reductions

$$\Omega_3 \to \Omega_3\omega_3 \to (\Omega_3\omega_3)\omega_3 \to \cdots \to ((((\Omega_3\omega_3)\omega_3)\omega_3)\omega_3)\omega_3 \to \cdots.$$

Then it is natural to consider the infinitary λ-term

$$\Omega_3^\infty := ((((\dots\omega_3)\omega_3)\omega_3)\omega_3)\omega_3 \quad \text{(infinitely many } \omega_3\text{'s)}$$

as the limit of the above sequence of reductions. In [5] Berarducci defines a new model of the lambda calculus that identifies two λ-terms if they have the same 'asympotic behaviour', namely they approach the same limit by repeated β-reductions. Such an idea is already present in the notion of Böhm tree. However, Böhm trees give no information on the inner structure of the unsolvable λ-terms, i.e., the Böhm tree of an unsolvable λ-term is defined to be $\bot$. Berarducci applies the idea of infinite unfolding also to the unsolvable λ-terms. For example, the infinite unfolding of the unsolvable λ-term Ω_3 is just the infinitary λ-term Ω_3^∞.

If A is an infinitary λ-term, $\mathrm{Var}(A)$ will denote the set of λ-variables x occurring either free/bound or as 'λx' in A. $\mathrm{Var}(A_1,\dots,A_n)$ will denote the set $\mathrm{Var}(A_1)\cup\cdots\cup\mathrm{Var}(A_n)$ for infinitary λ-terms $A_1,\dots,A_n$.

Recall that a complete partial order (*cpo*) is a partially ordered set $(X;\sqsubseteq)$ with least element $\bot$ (called bottom) such that every directed set $Y\subseteq X$ has a lub (least upper bound) $\bigsqcup Y$ (see [3, Chapter 1]). An element $x\in X$ is *compact* if, for every directed $Y\subseteq X$, $x\sqsubseteq\bigsqcup Y$ implies $x\sqsubseteq y$ for some $y\in Y$. A cpo $(X;\sqsubseteq)$ is *algebraic* if for all $x\in X$ the set $\{y\sqsubseteq x : y \text{ compact}\}$ is directed and $x=\bigsqcup\{y\sqsubseteq x : y \text{ compact}\}$. For $A,B\in F_I^\infty$ we let

$$A\sqsubseteq B \quad \text{iff } A \text{ results from } B \text{ by cutting off some subtrees.} \tag{3.1}$$

The pair $\langle F_I^\infty,\sqsubseteq\rangle$ constitutes an algebraic cpo, with A compact if, and only if, A is a finite tree.

We now provide a formal definition of the substitution operator for infinitary lambda terms. The definition is not immediate. For example, it is not clear how to define the substitution operator when $I=\{x_0,x_1,\dots,x_n,\dots\}$ is a countably infinite set of λ-variables and the substitution $[x_1 := (x_0(x_1(x_2(\dots))))]$ is applied to the λ-term $(\lambda x_0.x_1)$.

For every infinitary λ-term C and all λ-variables x,z with $z\notin\mathrm{Var}(C)$, denote by $C\{z/x\}$ the infinitary λ-term obtained as the result of the replacement of every free occurrence of x in C by z.

Assume I is a well-ordered infinite set of λ-variables. Let A, B, C be infinitary λ-terms over I such that $B\sqsubseteq C$. Let t be a λ-term such that $t\sqsubseteq A$. Define the substitution operator $t[x :=_{A,C} B]$ by induction over the complexity of the λ-term t as follows:

(i) $x[x :=_{x,C} B]=B$;

(ii) $y[x :=_{y,C} B]=y$ $(y\neq x)$;

(iii) $\bot[x :=_{A,C} B]=\bot$;

(iv) $(t_1t_2)[x :=_{A_1A_2,C} B]=(t_1[x :=_{A_1,C} B])(t_2[x :=_{A_2,C} B])$, where $t_1\sqsubseteq A_1$ and $t_2\sqsubseteq A_2$;

(v) $(\lambda x.t)[x :=_{\lambda x.A,C} B]=\lambda x.t$, where $t\sqsubseteq A$;

(vi) $(\lambda y.t)[x :=_{\lambda y.A,C} B]=\lambda y.t[x :=_{A,C} B]$ if $y\neq x$, y is not free in C and $t\sqsubseteq A$;

(vii) Let y be free in C, $y\neq x$, $t\sqsubseteq A$ and $I\backslash\mathrm{Var}(A,C)$ be nonempty. Let z be the first variable in $I\backslash\mathrm{Var}(A,C)$. Then we define

$$(\lambda y.t)[x :=_{\lambda y.A,C} B]=\lambda z.t\{z/y\}[x :=_{A\{z/y\},C} B]$$

(viii) $(\lambda y.t)[x :=_{\lambda y.A,C} B]=(\lambda xy.t)B$ if y is free in C, $y\neq x$, $t\sqsubseteq A$ and $I\backslash\mathrm{Var}(A,C)=\emptyset$.

The above definition is well given because $t\{z/y\}$ in item (vii) has the same complexity as t and $t\{z/y\} \sqsubseteq A\{z/y\}$ by the hypothesis $t \sqsubseteq A$.

If A is an infinitary λ-term, then the λ-term $A_{(n)}$ is obtained by cutting off the tree A at level n; in other words, $A_{(0)} = \bot$; if $A = BC$ then $A_{(n+1)} = B_{(n)}C_{(n)}$; if $A = \lambda x.B$ then $A_{(n+1)} = \lambda x.B_{(n)}$.

Extend the definition of substitution to the infinitary λ-terms A, B as follows:

$$A[x := B] = \bigsqcup_{n \geqslant 0} A_{(n)}[x :=_{A,B} B]. \tag{3.2}$$

The above definition is well given as proven in [40].

The following example shows that the substitution operator has sometimes an unexpected behaviour. Let $I = \{x_0, x_1, \ldots, x_n, \ldots\}$ be a countably infinite set of λ-variables, let $B = (x_0(x_1(x_2(\ldots))))$ be an infinitary λ-term and let $A = \lambda x_0.x_1$. Since $I \setminus \mathrm{Var}(A, B) = \emptyset$, then the infinitary λ-term $(\lambda x_1.A)B$ is the result of applying the substitution $[x_1 :=_{A,B} B]$ to the λ-term A.

The axioms of the infinitary lambda calculus are as follows: A, B, C and D are arbitrary infinitary λ-terms.

($\alpha\iota$) $\lambda x.A = \lambda y.A[x := y]$, for any λ-variable y that does not occur free in A;

($\beta\iota$) $(\lambda x.A)B = A[x := B]$;

(1) $A = A$;

(2) $A = B$ implies $B = A$;

(3) $A = B$, $B = C$ imply $A = C$;

(4) $A = B$, $C = D$ imply $AC = BD$;

(5) $A = B$ implies $\lambda x.A = \lambda x.B$.

An *infinitary lambda theory* is any set of equations between infinitary λ-terms that is closed under ($\alpha\iota$)- and ($\beta\iota$)-conversion and the five equality rules. The minimal infinitary lambda theory is denoted by $\lambda\beta\iota$. We also write $A = B \in \lambda\beta\iota$ as $A =_{\beta\iota} B$.

Let

$$\mathbf{F}_I^\infty := \langle F_I^\infty, \cdot^{\mathbf{F}_I^\infty}, \langle \lambda x^{\mathbf{F}_I^\infty} : x \in I\rangle, \langle x^{\mathbf{F}_I^\infty} : x \in I\rangle\rangle$$

be the absolutely free algebra of infinitary λ-terms. We denote by $\mathbf{F}_I^T$ the *term* algebra associated with the infinitary lambda theory T, that is, the quotient of $\mathbf{F}_I^\infty$ by T.

Theorem 19 (Salibra and Goldblatt [40]). *Let T be an infinitary lambda theory. Then the term algebra $\mathbf{F}_I^T$ of T is an* LAA_I.

We now prove a new result: a generalization of the Genericity Lemma [3, 14.3.24] to the infinitary lambda calculus. The notions of an unsolvable λ-term and of a normal form can be found in [3]. Recall that, for λ-terms w and s, $w =_\beta s$ means $\lambda\beta \vdash w = s$.

Theorem 20 (The Generalized Genericity Lemma). *Let t, u be finite λ-terms with t unsolvable and u having a normal form. Then for all λ-contexts $C(\xi)$ we have*

(i) $C(t) =_\beta u \Rightarrow C(w) =_\beta u$, *for all λ-terms w*;

(ii) $C(t) =_\beta u \Rightarrow \mathbf{A} \models C(\xi) = u$, *for all* LAA_I*s* $\mathbf{A}$;
(iii) $C(t) =_\beta u \Rightarrow C(B) =_{\beta\imath} u$, *for all infinitary* λ*-terms* B.

Proof. (i) is the Genericity Lemma in [3, 14.3.24].

(ii) From the hypothesis $C(t) =_\beta u$ and from (i) it follows that $C(w) =_\beta u$ for all λ-terms w. This means that the term algebra of the minimal lambda theory $\lambda\beta$ satisfies the identity $C(\xi) = u$. Since the variety of LAA_Is is generated by $\mathbf{F}_I^{\lambda\beta}$, the conclusion is immediate.

(iii) From Theorem 19 it follows that $\mathbf{F}_I^{\lambda\beta i}$ is an LAA_I. The conclusion follows now from (ii). $\square$

A consequence of Theorem 16 is that every infinitary lambda theory can be constructed from its restriction to the finite λ-terms by using some algebraic constructions.

Proposition 21. *Let* T *be an infinitary lambda theory and let* T_0 *be the restriction of* T *to the finitary lambda calculus, i.e.,* $T_0 = \{t = u \in T : t, u \in F_I\}$. *Then the term algebra* $\mathbf{F}_I^T$ *is a homomorphic image of a subalgebra of a Cartesian power of the term algebra* $\mathbf{F}_I^{T_0}$.

Proof. By Theorem 16, since the term algebra $\mathbf{F}_I^T$ is an element of the variety generated by the term algebra $\mathbf{F}_I^{T_0}$. $\square$

We now give an algebraic proof that the infinitary λ-calculus is consistent.

The *Böhm tree* of a λ-term t [3, Definition 10.1.4] is an infinitary λ-term $\mathrm{BT}(t)$ defined as follows:
(i) If t is unsolvable, then $\mathrm{BT}(t) = \perp$;
(ii) If t is solvable and $\lambda x_1 \cdots x_n.xt_1 \cdots t_k$ is the principal head normal form of t [3, Definition 8.3.20] then

$$\mathrm{BT}(t) = \lambda x_1 \cdots x_n.x\mathrm{BT}(t_1) \ldots \mathrm{BT}(t_k).$$

An infinitary λ-term D is a *Böhm-like tree* [3, Definition 10.1.12] if either $D = \perp$ or there exist λ-variables $x, x_1, \ldots, x_n$ and Böhm-like trees $D_1, \ldots, D_k$ such that $D = \lambda x_1 \cdots x_n.xD_1 \ldots D_k$. A Böhm-like tree is the Böhm tree of a λ-term if, and only if, the λ-variables occurring free in it are finite and it is recursively enumerable as a labeled tree [3, Theorem 10.1.23]. We follow Barendregt [2, p. 217] and identify all Böhm-like trees that differ only in the names of bound λ-variables. The best way to do this is to use the notation of de Bruijn explained in [7] and [3, Appendix C]. To keep matters readable we will write Böhm-like trees in the naive way.

Consider the following algebra in the similarity type of LAA_Is

$$\mathbf{BT} = \langle \mathsf{B}, \cdot^{\mathbf{BT}}, \lambda x^{\mathbf{BT}}, x^{\mathbf{BT}} \rangle_{x \in I},$$

where B is the set of Böhm-like trees over the infinite set I of λ-variables, and the operations are defined as follows [3, Definition 18.3.2], for all Böhm-like trees

A and B:
(i) $x^{\mathbf{BT}} = x$ for every $x \in I$;
(ii) $A \cdot^{\mathbf{BT}} B = \bigsqcup_{n \geqslant 0} \mathrm{BT}(A_{(n)} B_{(n)})$;
(iii) $\lambda x^{\mathbf{BT}}.A = \lambda x.A$.
The definition of $\cdot^{\mathbf{BT}}$ is well given by Lemma 18.3.3 in [3].

Let $\mathrm{BT} : F_I^{\infty} \to \mathsf{B}$ be the map defined by

$$\mathrm{BT}(A) = \bigsqcup_{n \geqslant 0} \mathrm{BT}(A_{(n)}), \quad \text{for all } A \in F_I^{\infty}.$$

BT is well defined by Lemma 14.3.12 in [3].

Lemma 22. *BT is a homomorphism from the absolutely free algebra* $\mathbf{F}_I^{\infty}$ *of infinitary* λ*-terms into the algebra* $\mathbf{BT}$ *of Böhm-like trees.*

Proof. It is sufficient to prove that $\mathrm{BT}(AB) = \mathrm{BT}(A) \cdot^{\mathbf{BT}} \mathrm{BT}(B)$. By definition of BT and by continuity of the application operator in $\mathbf{BT}$ [3, Proposition 18.3.4(ii)] we have $\mathrm{BT}(A) \cdot^{\mathbf{BT}} \mathrm{BT}(B) = \bigsqcup_{n \geqslant 0} \mathrm{BT}(A_{(n)}) \cdot^{\mathbf{BT}} \mathrm{BT}(B_{(n)})$. The conclusion is a consequence of $\mathrm{BT}(A_{(n)} B_{(n)}) = \mathrm{BT}(A_{(n)}) \cdot^{\mathbf{BT}} \mathrm{BT}(B_{(n)})$ [3, Proposition 18.3.4(i)]. □

Lemma 23. *The algebra* $\mathbf{BT}$ *is an* LAA_I.

Proof. The restriction $\mathrm{BT}_{|F_I}$ of the map BT to the set F_I of λ-terms is a homomorphism from the absolutely free algebra $\mathbf{F}_I$ of λ-terms into $\mathbf{BT}$. Since the kernel-relation θ of $\mathrm{BT}_{|F_I}$ is a lambda theory [3, Proposition 16.4.2], then by Lemma 8 and Proposition 9 we have that the quotient algebra $\mathbf{F}_I/\theta$ is a locally finite LAA_I. It follows that the subalgebra of $\mathbf{BT}$ determined by the set of Böhm trees is a locally finite LAA_I isomorphic to $\mathbf{F}_I/\theta$.

The validity of the axioms of LAA_I for arbitrary Böhm-like trees follows because every Böhm-like tree is the lub of a directed set of finite Böhm trees and the operations $\cdot^{\mathbf{BT}}$ and $\lambda x^{\mathbf{BT}}$ are continuous. □

Lemma 24. *If t and u are λ-terms such that $t \sqsubseteq A$ and $u \sqsubseteq B$, then*

$$\lambda\beta \vdash t[x :=_{A,B} u] = (\lambda x.t)u.$$

Proof. The proof is by induction on the structure of t. The only nontrivial case is when $t \equiv \lambda y.t_1$, $y \neq x$, y is free in B, $A = \lambda y.A_1$ and $I \setminus \mathrm{Var}(A,B)$ is not empty. Let z be the first λ-variable in $I \setminus \mathrm{Var}(A,B)$. Then we have

$$\begin{aligned} t[x :=_{A,B} u] &= \lambda z.t_1\{z/y\}[x :=_{A_1\{z/y\},B} u] \quad \text{[def. of substitution]} \\ &= \lambda z.(\lambda x.t_1\{z/y\})u \quad \text{[by induction]} \\ &= (\lambda xz.t_1\{z/y\})u \quad \text{[}(\beta)\text{-conversion, } z \text{ not free in } u\text{]} \\ &= (\lambda xy.t_1)u \quad \text{[}(\alpha)\text{-conversion, } z \text{ not free in } t_1\text{].} \end{aligned}$$

Let $\mathbf{A}$ be an LAA_I and let $\sqsubseteq$ be a partial order on the carrier set A of $\mathbf{A}$. We say that $\langle \mathbf{A}; \sqsubseteq \rangle$ is a *continuous* LAA_I if $(A; \sqsubseteq)$ is a cpo, $(\lambda x^{\mathbf{A}}.\bot)y = \bot$ for all $x, y \in I$, where $\bot$ is the bottom element, and the operations $\cdot^{\mathbf{A}}$ and $\lambda x^{\mathbf{A}}$ are continuous.

Lemma 25. *Let $\langle \mathbf{A}; \sqsubseteq \rangle$ be a continuous LAA_I and $f : F_I^{\infty} \to A$ be a homomorphism from the absolutely free algebra of infinitary λ-terms into $\mathbf{A}$. If $f(\bigsqcup Y) = \bigsqcup f(Y)$ for every directed set $Y \subseteq F_I^{\infty}$, then the relation-kernel θ_f of f (defined by $A\ \theta_f\ B$ iff $f(A) = f(B)$) is an infinitary lambda theory.*

Proof. The restriction of f to the (finitary) λ-terms is a homomorphism from the absolutely free algebra $\mathbf{F}_I$ of λ-terms (without constants) into $\mathbf{A}$. By Lemma 8 we have that the restriction of θ_f to the set of λ-terms is a lambda theory, i.e.,

$$\lambda\beta \vdash t = u \Rightarrow f(t) = f(u) \quad \text{for all } \lambda\text{-terms } t, u. \tag{3.3}$$

It is simple to verify by induction on the structure of the λ-term t that

$$t[x :=_{A,B} B] = \bigsqcup_{k \geqslant 0} t[x :=_{A,B} B_{(k)}]. \tag{3.4}$$

We now prove axiom $(\beta\iota)$.

$$\begin{aligned}
f(A[x := B]) &= f\left(\bigsqcup_n A_{(n)}[x :=_{A,B} B]\right) \quad [(3.2)]\\
&= f\left(\bigsqcup_{n,k} A_{(n)}[x :=_{A,B} B_{(k)}]\right) \quad [(3.4)]\\
&= \bigsqcup_{n,k} f(A_{(n)}[x :=_{A,B} B_{(k)}]) \quad [\text{continuity of } f]\\
&= \bigsqcup_{n,k} f((\lambda x.A_{(n)})B_{(k)}) \quad [(3.3) \text{ and Lemma 24}]\\
&= f\left(\bigsqcup_{n,k} (\lambda x.A_{(n)})B_{(k)}\right) \quad [\text{continuity of } f]\\
&= f((\lambda x.A)B).
\end{aligned}$$

Here is the proof of axiom $(\alpha\iota)$.

$$\begin{aligned}
f(\lambda y.A[x := y]) &= f(\lambda y.(\lambda x.A)y) \quad [(\beta\iota)]\\
&= f\left(\bigsqcup_n \lambda y.(\lambda x.A_{(n)})y\right)\\
&= \bigsqcup_n f(\lambda y.(\lambda x.A_{(n)})y) \quad [\text{continuity of } f]\\
&= \bigsqcup_n f(\lambda x.A_{(n)}) \quad [(\alpha)\text{-conversion, } y \text{ not free in } A_{(n)}]
\end{aligned}$$

$$= f\left(\bigsqcup_n \lambda x.A_{(n)}\right) \quad \text{[continuity of } f\text{]}$$

$$= f(\lambda x.A). \qquad \square$$

Theorem 26. *The infinitary lambda calculus is consistent.*

Proof. By Lemma 25 applied to the homomorphism BT we have that the relation-kernel θ_{BT} of BT is an infinitary lambda theory. Moreover, θ_{BT} is not trivial because two distinct Böhm trees are not θ_{BT}-equivalent. $\square$

4. The combinatory models of lambda calculus

This section has a survey character. We summarize definitions and results concerning the relationships between the lambda abstraction algebras and the combinatory models of lambda calculus. Our main reference will be [37]. Some of the results reviewed in this section will be used in the next one, where we study the idempotent expansions of LAAs. We also provide new easy proofs of some results appeared in [40]. The only new result of this section is Proposition 29.

We think that a careful reading of this section will make more understandable the theory of LAAs.

4.1. The variety of combinatory algebras

Combinatory logic is a formalism for writing expressions which denote functions. Combinators are designed to perform the same tasks as λ-terms, but without using bound variables. As an informal example, consider the expression xx which represents the generic function x applied to itself. While in lambda calculus one can construct the λ-term $\lambda x.xx$ denoting the function whose values are given by the expression xx, in combinatory logic one can construct the same function by introducing a new symbol (combinator), for example C, and defining it as $Cx = xx$. Schönfinkel and Curry discovered that a formal system of combinators having the same expressive power of lambda calculus can be based on only two primitive combinators.

We begin with the definition of a basic notion in combinatory logic and lambda calculus.

Definition 27 (*Curry* [12] *and Schönfinkel* [41]). Let $\mathbf{C} = \langle C, \cdot^{\mathbf{C}}, \mathbf{k}^{\mathbf{C}}, \mathbf{s}^{\mathbf{C}} \rangle$ be an algebra where $\cdot^{\mathbf{C}}$ is a binary operation and $\mathbf{k}^{\mathbf{C}}, \mathbf{s}^{\mathbf{C}}$ are constants. $\mathbf{C}$ is a *combinatory algebra* if it satisfies the following identities (as usual the symbol $\cdot$ and the superscript $^{\mathbf{C}}$ are omitted, and association is to the left):

$$\mathbf{k}xy = x; \quad \mathbf{s}xyz = xz(yz). \tag{4.1}$$

The class of combinatory algebras, denoted by CA, forms a variety of algebras. Combinatory logic **CL** is the equational theory axiomatized by identities (4.1).

k and **s** are called *combinators*. In the equational language of combinatory algebras the derived combinators **i** and **1** are defined as follows: $\mathbf{i} := \mathbf{skk}$ and $\mathbf{1} := \mathbf{s}(\mathbf{ki})$, and note that every combinatory algebra satisfies the identities $\mathbf{i}x = x$ and $\mathbf{1}xy = xy$.

Suitable reducts of arbitrary LAAs turn out to be combinatory algebras. Let $\mathbf{A}$ be an LAA_I. By the *combinatory reduct* of $\mathbf{A}$ we mean the algebra

$$\mathrm{Cr}\ \mathbf{A} = \langle A, \cdot^{\mathbf{A}}, \mathbf{k}^{\mathbf{A}}, \mathbf{s}^{\mathbf{A}} \rangle,$$

where

$$\mathbf{k}^{\mathbf{A}} = (\lambda xy.x)^{\mathbf{A}} \quad \text{and} \quad \mathbf{s}^{\mathbf{A}} = (\lambda xyz.xz(yz))^{\mathbf{A}}. \tag{4.2}$$

The λ-variables x, y, and z are assumed to be distinct. In Proposition 29 below we show that the definition of $\mathbf{k}^{\mathbf{A}}$ and $\mathbf{s}^{\mathbf{A}}$ in (4.2) is independent of the choice of $x, y, z \in I$ if $|I| \geqslant 7$. This result improves the limit $|I| \geqslant 9$ obtained as a consequence of Proposition 4.5 in [35].

We start with a lemma. Recall that $I^\star$ is the set of all finite sequences of elements of I without repetitions. If $\bar{y} = y_1 \cdots y_n$ and $\bar{x} = x_1 \cdots x_n$ are sequences of λ-variables and t is a λ-term, then $t[\bar{x}/\bar{y}]$ is an abbreviation for $t[x_1/y_1] \cdots [x_n/y_n]$.

Lemma 28. *Let I be a possibly finite set and let $\bar{y}, \bar{x} \in I^\star$ be sequences of the same length such that $\bar{x} \cap \bar{y} = \emptyset$. If $t \in F_I$ is a λ-term and no x_i occurs either free/bound or as 'λx_i' in t, then we have*

$$\mathsf{LAA}_I \models \lambda \bar{y}.t = \lambda \bar{x}.t[\bar{x}/\bar{y}]. \tag{4.3}$$

(*Notice that simple substitution is sufficient in* (4.3) *because each x_i is free for y_i in t.*)

Proof. An easy induction on the structure of t gives

$$(\lambda y.t)x = t[x/y] \tag{4.4}$$

for all distinct λ-variables x, y such that x is fresh with respect to t.

The proof of (4.3) is by induction on the length n of the sequence $\bar{y}$. If $n = 0$ there is nothing to show. Assume $n > 0$, $\bar{y} = y_1\bar{y}'$ and $\bar{x} = x_1\bar{x}'$.

$$\begin{aligned}
\lambda \bar{y}.t &= \lambda y_1.\lambda \bar{y}'.t \\
&= \lambda y_1.\lambda \bar{x}'.t[\bar{x}'/\bar{y}'] \quad \text{[by induction]} \\
&= \lambda x_1.(\lambda y_1.\lambda \bar{x}'.t[\bar{x}'/\bar{y}'])x_1 \quad [(\alpha),\ x_1 \text{ does not occur in } t[\bar{x}'/\bar{y}']] \\
&= \lambda x_1.\lambda \bar{x}'.(\lambda y_1.t[\bar{x}'/\bar{y}'])x_1) \quad [(\beta_6)] \\
&= \lambda \bar{x}.t[\bar{x}'/\bar{y}'][x_1/y_1] \quad [(4.4)] \\
&= \lambda \bar{x}.t[x_1/y_1][\bar{x}'/\bar{y}'] \quad [\bar{x} \text{ fresh}] \\
&= \lambda \bar{x}.t[\bar{x}/\bar{y}]. \qquad \square
\end{aligned}$$

Proposition 29. *Let* $\mathbf{A}$ *be an* LAA_I. *The definition of* $\mathbf{k}^{\mathbf{A}}$ *and* $\mathbf{s}^{\mathbf{A}}$ *in* (4.2) *is independent of the choice of* $x, y, z \in I$ *as soon as* $|I| \geqslant 7$.

Proof. Assume $I = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. Let $\mathbf{s} = \lambda x_1 x_2 x_3 . x_1 x_3 (x_2 x_3)$ and let $x', y', z' \in I$ be three distinct λ-variables. Let $K = \{x_1, x_2, x_3\}$ and $K' = \{x', y', z'\}$. If $K \cap K' = \emptyset$ the conclusion $\mathbf{s} = \lambda x' y' z' . x' z' (y' z')$ follows from Lemma 28.

If $K \cup K'$ has at most 4 distinct elements, we consider other three distinct λ-variables $q, w, v \in I \setminus (K \cup K')$. We have the conclusion by applying two times Lemma 28: $\lambda x_1 x_2 x_3 . x_1 x_3 (x_2 x_3) = \lambda q w v . q v (w v) = \lambda x' y' z' . x' z' (y' z')$.

Assume that $K \cup K'$ has exactly 5 distinct elements. Without loss of generality, let $K' = \{x_3, x_4, x_5\}$. We have the conclusion by applying three times Lemma 28: $\lambda x_1 x_2 x_3 . x_1 x_3 (x_2 x_3) = \lambda x_4 x_5 x_6 . x_4 x_6 (x_5 x_6) = \lambda x_1 x_2 x_7 . x_1 x_7 (x_2 x_7) = \lambda x_3 x_4 x_5 . x_3 x_5 (x_4 x_5)$.

A similar argument works for $\mathbf{k}$. □

In the sequel, we will assume $|I| \geqslant 7$ unless otherwise specified.

A subalgebra of the combinatory reduct of an LAA_I $\mathbf{A}$ (i.e., a subset of $\mathbf{A}$ containing $\mathbf{k}^{\mathbf{A}}$ and $\mathbf{s}^{\mathbf{A}}$ and closed under $\cdot^{\mathbf{A}}$) is called a *combinatory subreduct* of $\mathbf{A}$. The *zero-dimensional subreduct* of $\mathbf{A}$ is the combinatory subreduct

$$\mathbf{Zd}\,\mathbf{A} = \langle \mathrm{Zd}\ \mathbf{A}, \cdot^{\mathbf{A}}, \mathbf{k}^{\mathbf{A}}, \mathbf{s}^{\mathbf{A}} \rangle,$$

where $\mathrm{Zd}\ \mathbf{A} = \{ a \in A : \boldsymbol{\Delta}^{\mathbf{A}} a = \emptyset \}$, the set of zero-dimensional elements of $\mathbf{A}$.

Theorem 30 (Salibra and Goldblatt [40]). *The combinatory reduct* Cr $\mathbf{A}$ *of an infinite dimensional* LAA_I $\mathbf{A}$ *is a combinatory algebra.*

Proof. Let $\mathbf{k} = \lambda x y . x$ and $\mathbf{s} = \lambda x y z . x z (y z)$. The term algebra $\mathbf{F}_I^{\lambda\beta}$ of the minimal lambda theory $\lambda\beta$ generates the variety LAA_I and satisfies the identities $\mathbf{k}\xi\mu = \xi$ and $\mathbf{s}\xi\mu\nu = \xi\nu(\mu\nu)$. □

In the equational logic of combinatory algebras it is traditional to let λ-variables play the role of real variables. We follow this convention in the next definition. Recall that x, y, z, possibly with subscripts, denote arbitrary distinct λ-variables in I. By a *combinatory term* we mean a term of the equational logic of combinatory algebras in the usual sense. Thus $\mathbf{k}$, $\mathbf{s}$, and x, for every λ-variable x, are combinatory terms. If s and t are combinatory terms, so is st. A combinatory term is *closed* (or *ground*) if it contains no λ-variables. Note that context variables do not occur in combinatory terms. Let $\mathbf{C}$ be a combinatory algebra. Let $\bar{c}$ be a new symbol for each $c \in C$. Extend the language of combinatory algebras by adjoining $\bar{c}$ as a new constant symbol for each $c \in C$. A term t in this extended language is called a *combinatory polynomial over* $\mathbf{C}$. The set of all such polynomials is denoted by $P_I(C)$. If $t = t(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ includes all the λ-variables occurring in t, and $v_1, \dots, v_n \in C$, then $t^{\mathbf{C}}(v_1, \dots, v_n)$ will denote the value of t in $\mathbf{C}$ when x_i is interpreted as v_i and each new constant $\bar{c}$ as c.

The following result is well known (see [29, 12] and [3, Theorem 5.1.10]).

Proposition 31 (Combinatory completeness lemma). *Let* $\mathbf{C}$ *be a combinatory algebra and let* $t(x_1,\dots,x_n)$ *be a combinatory polynomial over* $\mathbf{C}$ *whose variables all occur in the list* $x_1,\dots,x_n$. *Then there exists an element* c *in* $\mathbf{C}$ *such that, for all* $v_1,\dots,v_n \in C$,

$$t^{\mathbf{C}}(v_1,\dots,v_n) = cv_1\cdots v_n.$$

The combinatory completeness lemma depends on the following definition and lemma that shows that some aspects of lambda abstraction can be simulated in combinatory algebras.

Let $\mathbf{C}$ be a combinatory algebra. For each λ-variable x define a transformation λ^*x of the set $P_I(C)$ of combinatory polynomials over $\mathbf{C}$ as follows: $\lambda^*x(x) = \mathbf{i}$. Let t be a combinatory polynomial different from x. If x does not occur in t, define $\lambda^*x(t) = \mathbf{k}t$; in particular, $\lambda^*x(\bar{v}) = \mathbf{k}\bar{v}$ for every $v \in C$. Otherwise, t must be of the form sr where s and r are combinatory polynomials, at least one of which contains x; in this case define $\lambda^*x(t) = \mathbf{s}(\lambda^*x(r))\,(\lambda^*x(s))$.

Lemma 32. *Let* $\mathbf{C}$ *be a combinatory algebra,* t *a combinatory polynomial over* $\mathbf{C}$, *and* x *a* λ-*variable. Let* $y_1,\dots,y_n$ *be any list of* λ-*variables that includes all* λ-*variables occurring in* t *except* x, *and write* $t = t(x, y_1,\dots,y_n)$ *and* $\lambda^*x(t) = (\lambda^*x(t))(y_1,\dots,y_n)$. *Then for all* $v, u_1,\dots,u_n \in C$,

$$t^{\mathbf{C}}(v,u_1,\dots,u_n) = ((\lambda^*x(t))^{\mathbf{C}}(u_1,\dots,u_n))v,$$

i.e., the combinatory algebra $\mathbf{C}$ *satisfies the equation* $(\lambda^*x(t))x = t$; *in symbols,*

$$\mathbf{C} \models (\lambda^*x(t))x = t.$$

λ^*x is an operation on combinatory polynomials; it does not define directly an operator on combinatory algebras. It can be used to define translations $\mathrm{CL} : F_I(C) \to P_I(C)$ and $\lambda : P_I(C) \to F_I(C)$ from lambda terms to combinatory polynomials and vice versa [3, Definition 7.3.1]:

$$x_\lambda = x,$$

$$c_\lambda = c,$$

$$\mathbf{k}_\lambda = \lambda xy.x,$$

$$\mathbf{s}_\lambda = \lambda xyz.xz(yz),$$

$$(tu)_\lambda = t_\lambda u_\lambda,$$

$$x_{\mathrm{CL}} = x,$$

$$c_{\mathrm{CL}} = c,$$

$$(tu)_{\mathrm{CL}} = t_{\mathrm{CL}}u_{\mathrm{CL}},$$

$$(\lambda x.t)_{\mathrm{CL}} = \lambda^*x(t_{\mathrm{CL}}).$$

Combinatory logic is weaker than lambda calculus; for example, $\mathbf{s}_\lambda x$, $\mathbf{s}_\lambda xy$ and $\mathbf{k}_\lambda x$ are not normal forms in the lambda calculus, while $\mathbf{s}x$, $\mathbf{s}xy$ and $\mathbf{k}x$ are normal forms in combinatory logic. The best we can obtain by these translations is summarized in the following proposition [3, Lemma 7.3.3, Theorem 7.3.10(i)]:

Proposition 33. *Let* CA *be the variety of combinatory algebras*:

(i) $\mathsf{CA} \models t = u \Rightarrow \lambda\beta \vdash t_\lambda = u_\lambda$, *for all combinatory terms* t, u, *while the converse is not true. For example*, $\lambda\beta \vdash \mathbf{s}_\lambda \mathbf{k}_\lambda = \mathbf{k}_\lambda \mathbf{i}_\lambda$ *while* $\mathsf{CA} \not\models \mathbf{sk} = \mathbf{ki}$.

(ii) $\lambda\beta \vdash t = u \Leftarrow \mathsf{CA} \models t_{\mathrm{CL}} = u_{\mathrm{CL}}$, *for all* λ-*terms* t, u, *while the converse is not true. Indeed, the set* $\{t = u : \mathsf{CA} \models t_{\mathrm{CL}} = u_{\mathrm{CL}}\}$ *does not constitute a lambda theory because the following equality rule fails for combinatory logic*: $\mathsf{CA} \models t = u \Rightarrow \mathsf{CA} \models \lambda^* x(t) = \lambda^* x(u)$.

(iii) $\lambda\beta \vdash t_{\mathrm{CL},\lambda} = t$, *for every* λ-*term* t.

(iv) *There exists a combinatory term* t *such that* $\mathsf{CA} \not\models t_{\lambda,\mathrm{CL}} = t$. *For example*, $\mathsf{CA} \not\models \mathbf{k} = \mathbf{k}_{\lambda,\mathrm{CL}}$.

Let $t = t(x_1, \ldots, x_n)$ be a combinatory term, where the sequence $x_1, \ldots, x_n$ includes all the λ-variables occurring in t. The role played by the λ-term t_λ in the theory of LAA_Is is different from the corresponding one played by t in the theory of combinatory algebras. The reason is that t_λ is a λ-context without context variables (i.e., algebraic variables) since the λ-variables $x_1, \ldots, x_n$ are nullary operations in the similarity type of lambda abstraction algebras. Instead, the λ-variables $x_1 \cdots x_n$ in t play the role of real algebraic variables. Thus, for every LAA_I $\mathbf{A}$, $t_\lambda^{\mathbf{A}}$ will denote the value of t_λ in $\mathbf{A}$ when x_i is interpreted as $x_i^{\mathbf{A}}$, while $t^{\mathrm{Cr}\,\mathbf{A}}$ has a different interpretation: $t^{\mathrm{Cr}\,\mathbf{A}}$ is a function from A^n into A. More precisely, if $a_1, \ldots, a_n \in A$, then $t^{\mathrm{Cr}\,\mathbf{A}}(a_1, \ldots, a_n)$ will denote the value of t in Cr $\mathbf{A}$ when x_i is interpreted as a_i. Thus, we have

$$t_\lambda^{\mathbf{A}} = t^{\mathrm{Cr}\,\mathbf{A}}(x_1^{\mathbf{A}}, \ldots, x_n^{\mathbf{A}}). \tag{4.5}$$

4.2. The variety of λ-algebras

Those combinatory algebras for which the combinatory polynomial transformation λx^* simulates lambda abstraction form a variety. They are called *λ-algebras*; the concept is essentially due to Curry. The zero-dimensional subreduct of a LFA_I is a λ-algebra, while every λ-algebra is isomorphic to a zero-dimensional subreduct of an LFA_I (Corollary 40). This leads to a categorical equivalence between the category of λ-algebras and the category of LFA_Is (Theorem 41). Moreover, the free extension of a λ-algebra $\mathbf{C}$ (by a set I) in the variety of combinatory algebras can be turned into a LFA_I whose zero-dimensional subreduct is $\mathbf{C}$.

Definition 34. A combinatory algebra $\mathbf{C}$ is a *λ-algebra* if it satisfies the following condition for all combinatory terms t, u:

$$\lambda\beta \vdash t_\lambda = u_\lambda \Rightarrow \mathbf{C} \models t = u.$$

The hypothesis that t, u range over the set of combinatory terms and not over the set $P_I(C)$ of combinatory polynomials, as in the standard definition in Barendregt's book, is not restrictive (see [42] for a simple proof of this fact).

A simple proof of the following proposition is due to Selinger [43].

Proposition 35. *The class of λ-algebras, denoted by* LA, *forms a variety axiomatized by the defining identities of combinatory algebras and by the identities* $t = u$ *between closed combinatory terms* (*no variables are involved*) *such that* $\lambda\beta \vdash t_\lambda = u_\lambda$.

Curry discovered that only a finite number of identities between closed combinatory terms are sufficient for axiomatizing λ-algebras over combinatory algebras (see [3, Chapter 7]).

We know in general that the combinatory reduct of every LAA_I is a λ-algebra. This was proven in [40].

Theorem 36 (Salibra and Goldblatt [40]). *The combinatory reduct of an infinite dimensional* LAA_I *is a λ-algebra.*

Proof. Let $\mathbf{A}$ be an LAA_I. The combinatory reduct of $\mathbf{A}$ is a combinatory algebra from Theorem 30. Let t, u be *closed* combinatory terms. Then we have

$$\begin{aligned} \lambda\beta \vdash t_\lambda = u_\lambda &\Rightarrow \mathbf{A} \models t_\lambda = u_\lambda \quad \text{[Proposition 11]} \\ &\Leftrightarrow t_\lambda^{\mathbf{A}} = u_\lambda^{\mathbf{A}} \\ &\Leftrightarrow t^{\mathrm{Cr}\,\mathbf{A}} = u^{\mathrm{Cr}\,\mathbf{A}} \quad \text{[(4.5) and } t, u \text{ closed]} \\ &\Leftrightarrow \mathrm{Cr}\,\mathbf{A} \models t = u \end{aligned}$$

The conclusion follows from Proposition 35. □

The axioms of a λ-algebra are designed expressly to prove the next lemma. We require a definition.

Let $\mathbf{C}$ be a combinatory algebra. Recall that $P_I(C)$ is the set of combinatory polynomials over $\mathbf{C}$. Recall also that the members of $P_I(C)$ are constructed from λ-variables in I and constant symbols $\mathbf{k}$, $\mathbf{s}$, and $\bar{c}$ for all elements c of $\mathbf{C}$.

Let $D_{\mathbf{C}}$ be the *equational diagram* of $\mathbf{C}$, i.e., the set of all equations of the form $\bar{c}\bar{d} = \bar{e}$ for $c, d, e \in C$ such that $cd = e$; we also include the two equations $\mathbf{k} = \bar{c}$ and $\mathbf{s} = \bar{d}$, where $c = \mathbf{k}^{\mathbf{C}}$ and $d = \mathbf{s}^{\mathbf{C}}$. Let $\equiv_{\mathbf{C}}$ be the equivalence relation of $P_I(C)$ such that $t \equiv_{\mathbf{C}} s$ iff the equation $t = s$ is a logical consequence of $D_{\mathbf{C}}$ together with the axioms of combinatory logic.

Lemma 37. *Let* $\mathbf{C}$ *be a λ-algebra and let* t, s *be combinatory polynomials over* $\mathbf{C}$. *Then* $t \equiv_{\mathbf{C}} s$ *if and only if* $\lambda^* x(t) \equiv_{\mathbf{C}} \lambda^* x(s)$ *for every* $x \in I$.

The proof of the above lemma can be found in [29, Lemma 7.12]. A remarkable algebraic and simple proof was discovered by Krivine [26]. It is outlined at the beginning of the last section in this paper.

Only identities between closed combinatory terms are sufficient for axiomatizing λ-algebras over combinatory algebras; hence, if $\mathbf{C}$ is a λ-algebra and $\mathsf{LA} \models t = u$, then we have $t \equiv_{\mathbf{C}} u$.

The following well-known result shows that lambda calculus is equivalent to the equational theory of the variety of λ-algebras [3, Theorem 7.3.10].

Proposition 38. (i) $\lambda\beta \vdash t = u \Leftrightarrow \mathsf{LA} \models t_{CL} = u_{CL}$, *for all* λ-*terms* t, u.
(ii) $\mathsf{LA} \models t_{\lambda,CL} = t$, *for every combinatory term* t.
(iii) $\mathsf{LA} \models t = u \Leftrightarrow \lambda\beta \vdash t_\lambda = u_\lambda$, *for all combinatory terms* t, u.

Condition (iii) is an easy consequence of (i) and (ii).

We denote by $\mathbf{C}[I]$ the *free extension of* $\mathbf{C}$ *by* I in the variety of combinatory algebras. $\mathbf{C}[I]$ is an expansion of $\mathbf{C}$ defined up to isomorphism by the following universal mapping properties: ($C[I]$ is the universe of $\mathbf{C}[I]$.) (1) $I \subseteq C[I]$; (2) $\mathbf{C}[I]$ is a combinatory algebra; (3) for every homomorphism $h : \mathbf{C} \to \mathbf{A}$ from $\mathbf{C}$ into a combinatory algebra $\mathbf{A}$ and every mapping $g : I \to A$ there exists a unique homomorphism $f : \mathbf{C}[I] \to \mathbf{A}$ extending both h and g. A concrete description of $\mathbf{C}[I]$ as a quotient of the algebra of the combinatory polynomials over C (with λ-variables from I) by the congruence $\equiv_{\mathbf{C}}$ (defined before Lemma 37) may be found in [29, p. 109]. Let t be a combinatory polynomial over $\mathbf{C}$. $t^{\mathbf{C}[I]}$ denotes the unique interpretation of t in $\mathbf{C}[I]$ when each variable x in t is interpreted as $x^{\mathbf{C}[I]}$, each constant $\bar{c}$ as c, and the combinators $\mathbf{k}, \mathbf{s}$ as $\mathbf{k}^{\mathbf{C}}, \mathbf{s}^{\mathbf{C}}$. It follows easily from basic principles of universal algebra that $t^{\mathbf{C}[I]} = u^{\mathbf{C}[I]}$ iff $t \equiv_{\mathbf{C}} u$, so that $t^{\mathbf{C}[I]}$ denotes the equivalence class of t in the concrete description of $\mathbf{C}[I]$ as a quotient.

We define λ-abstractions $\lambda x^{\mathbf{C}[I]}$ on $\mathbf{C}[I]$ for all $x \in I$ as follows: Let $a \in C[I]$. Choose any $t \in P(C)$ such that $t^{\mathbf{C}[I]} = a$. Define

$$\lambda x^{\mathbf{C}[I]}.a = (\lambda^* x(t))^{\mathbf{C}[I]}.$$

Lemma 37 guarantees $\lambda x^{\mathbf{C}[I]}$ is well defined. The algebra obtained by adjoining these operations will also be denoted by $\mathbf{C}[I]$.

Theorem 39 (Pigozzi and Salibra [37, Theorem 3.1]). *Let* $\mathbf{C}$ *be a* λ-*algebra.* $\mathbf{C}[I]$ *is an* LFA_I *whose zero-dimensional subreduct is* $\mathbf{C}$. *Moreover, it is universal with respect to this property in the sense that, if* $h : \mathbf{C} \to \mathbf{Zd}\,\mathbf{A}$ *is any homomorphism of* $\mathbf{C}$ *into the zero-dimensional subreduct of an* LAA_I $\mathbf{A}$, *then* h *extends uniquely to a homomorphism* h_I *from* $\mathbf{C}[I]$ *into* $\mathbf{A}$.

The following is a consequence of the above theorem and of Theorem 36.

Corollary 40 (Pigozzi and Salibra [37, Corollary 3.1]). *Let* $\mathbf{C}$ *be a combinatory algebra. The following are equivalent*:
(i) $\mathbf{C}$ *is a* λ-*algebra*;
(ii) $\mathbf{C} = \mathbf{Zd}\,\mathbf{A}$ *for an* LAA_I $\mathbf{A}$;

(iii) $\mathbf{C}$ *is a subalgebra of* $\mathbf{Zd}\,\mathbf{A}$ *for an* LAA_I $\mathbf{A}$;
(iv) $\mathbf{C}$ *is a combinatory subreduct of a locally finite* LAA_I;
(v) $\mathbf{C}$ *is a combinatory subreduct of an* LAA_I.

The categories of λ-algebras and of LFA_Is are equivalent.

Theorem 41 (Pigozzi and Salibra [37, Theorem 3.2]). *The category of λ-algebras and the category of* LFA_I*s* (*with I infinite*) *are equivalent. Then, for any* LFA_I*s* $\mathbf{A}$ *and* $\mathbf{B}$, *any combinatory homomorphism h from* $\mathbf{Zd}\,\mathbf{A}$ *to* $\mathbf{Zd}\,\mathbf{B}$ *extends uniquely to a lambda abstraction algebra homomorphism from* $\mathbf{A}$ *to* $\mathbf{B}$. *Moreover, if h is one–one and/or onto, so is its extension. Thus,* $\mathbf{A}$ *and* $\mathbf{B}$ *are isomorphic if their zero-dimensional subreducts are isomorphic.*

4.3. Lambda models

Lambda models were introduced by Meyer [29] as an alternative first-order characterization of environment models. In fact, they form an elementary class, while the definition of environment model is higher order.

Rich LAAs, which we will define soon, correspond to lambda models in the same way that all LAAs correspond to λ-algebras, the zero-dimensional subreduct of a rich LFA_I is a lambda model and vice versa. This leads to a categorical equivalence similar to the one for λ-algebras given in Theorem 41. The free extension of a lambda model $\mathbf{C}$ (by a set I) in the variety *generated by* $\mathbf{C}$ can be turned into a LFA_I whose zero-dimensional subreduct is $\mathbf{C}$. This result is the basis for characterizing lambda models as those λ-algebras $\mathbf{C}$ whose free extensions in the variety of combinatory algebras and in the variety generated by $\mathbf{C}$ coincide.

Definition 42 (*Meyer* [29] *and Scott* [45]). A *lambda model* is a λ-algebra $\mathbf{C}$ satisfying the following condition, for all $u, w \in C$:

$$\text{If } uv = wv \quad \text{for all } v \in C, \text{ then } \mathbf{1}u = \mathbf{1}w.$$

Condition (i) is called the *Meyer–Scott axiom*. In the first-order language of combinatory algebras it takes the following form:

$$\forall x \forall y (\forall z (xz = yz) \Rightarrow \mathbf{1}x = \mathbf{1}y).$$

The particular form of the definition of lambda model given in Definition 42 is taken from Barendregt [3, Definition 5.2.7].

The following result is Proposition 5.2.9 in [3].

Proposition 43. *A λ-algebra* $\mathbf{C}$ *is a lambda model if and only if it satisfies the following condition, for all combinatory polynomials $t, u \in P_I(C)$*:

$$\mathbf{C} \models t = u \Rightarrow \mathbf{C} \models \lambda^* x(t) = \lambda^* x(u).$$

We do not know in general if the combinatory reduct of every LAA_I is a lambda model. We conjecture that this is true. We have shown in [37, Proposition 4.2] that the combinatory reduct of every locally finite LAA_I is a lambda model. The same proof can be extended without any change to the dimension-complemented case; we recall from [35] that an LAA_I $\mathbf{A}$ is *dimension-complemented* if the dimension set I is infinite and there is no element in A whose dimension set is all I. (This includes all the locally finite LAA_Is.)

Let $\mathbf{C}$ be a lambda model. We denote by $\mathbf{C}^*[I]$ the *free extension of* $\mathbf{C}$ *by* I in the variety *generated by* $\mathbf{C}$. Let $t \in P_I(\mathbf{C})$, i.e., a combinatory polynomial over $\mathbf{C}$. $t^{\mathbf{C}^*[I]}$ denotes the unique interpretation of t in $\mathbf{C}^*[I]$ when each variable x in t is interpreted as $x^{\mathbf{C}^*[I]}$, each constant $\bar{c}$ as c, and the combinators $\mathbf{k}, \mathbf{s}$ as $\mathbf{k}^{\mathbf{C}}, \mathbf{s}^{\mathbf{C}}$. It follows easily from the basic principles of universal algebra that $t^{\mathbf{C}^*[I]} = u^{\mathbf{C}^*[I]}$ iff $\mathbf{C} \models t = u$.

We define λ-abstractions $\lambda x^{\mathbf{C}^*[I]}$ on $\mathbf{C}^*[I]$ as follows: Let $a \in C^*[I]$. Choose any $t \in P_I(\mathbf{C})$ such that $t^{\mathbf{C}^*[I]} = a$. Then we define

$$\lambda x^{\mathbf{C}^*[I]}.a = (\lambda^* x(t))^{\mathbf{C}^*[I]}.$$

The algebra obtained by adjoining these new operations $\lambda x^{\mathbf{C}^*[I]}$ will also be denoted by $\mathbf{C}^*[I]$.

Theorem 44 (Pigozzi and Salibra [37, Theorem 4.1]). *Let* $\mathbf{C}$ *be a lambda model. Then* $\mathbf{C}^*[I]$ *is a* LFA_I *whose zero-dimensional subreduct is* $\mathbf{C}$.

By the equivalence between the categories of λ-algebras and LFA_Is it follows immediately that the free extension of a lambda model $\mathbf{C}$ by I in the variety of combinatory algebras is equal to the free extension of $\mathbf{C}$ by I in the variety generated by $\mathbf{C}$.

A lambda theory T is closed under the term rule [3, Definition 4.1.10] if the following property holds:

$$(\mathbf{tr})\ tu = su \in T \quad \text{for every closed term } u \Rightarrow tx = sx \in T, \text{ for every } \lambda\text{-variable } x.$$

Notice that the term rule can be expressed in the following equivalent way. For every λ-variable x:

$$(\lambda x.t)u = (\lambda x.s)u \in T \quad \text{for every closed term } u \Rightarrow t = s \in T.$$

The following is the algebraic version of the term rule.

Definition 45 (*Pigozzi and Salibra* [37, *Definition* 4.2]). An LAA_I $\mathbf{A}$ is *rich* if, for all finite-dimensional elements $a, b \in A$ and all $x \in I$, we have that

$$(\forall c \in \mathrm{Zd}\,\mathbf{A} : (\lambda x.a)c = (\lambda x.b)c) \Rightarrow a = b.$$

Rich LAA_Is correspond roughly to rich polyadic Boolean algebras [20].

Let $\mathscr{V}$ be an arbitrary variety of algebras and $\mathbf{A} \in \mathscr{V}$. We recall that $\mathbf{A}$ is generic in $\mathscr{V}$ if an identity holds in $\mathbf{A}$ iff it holds in $\mathscr{V}$ (see [19, p. 383]).

We recall that **CL** is the equational theory axiomatized by the axioms $\mathbf{k}xy = x$ and $\mathbf{s}xyz = xz(yz)$ of combinatory logic, and, for every combinatory algebra **C**, $D_{\mathbf{C}}$ is the equational diagram of **C**.

Theorem 46 (Pigozzi and Salibra [37, Theorem 4.2]). *Let* **C** *be a combinatory algebra. The following are equivalent*:
 (i) **C** *is a lambda model*;
 (ii) $\mathbf{C} = \mathbf{Zd}\,\mathbf{A}$ *for a rich* LFA_I **A**.
 (iii) $\mathbf{C} = \mathbf{Zd}\,\mathbf{A}$ *for a rich* LAA_I **A**.
 (iv) **C** *is a* λ*-algebra and the free extension of* **C** *by* I *in the variety of combinatory algebras is equal to the free extension of* **C** *by* I *in the variety generated by* **C**.
 (v) **C** *is generic in the variety axiomatized by* $\mathbf{CL} + D_{\mathbf{C}}$.

In view of the equivalence of (i) and (ii) in Theorem 46 it is easy to see that an LAA_I **A** is rich iff the zero-dimensional subreduct $\mathbf{Zd}\,\mathbf{A}$ of **A** is a lambda model. Thus the following corollary is an immediate consequence of the equivalence of the categories of λ-algebras and LFA_Is (Theorem 41).

Corollary 47 (Pigozzi and Salibra [37, Corollary 4.1]). *The categories of lambda models and rich* LFA_I*s are equivalent.*

5. Idempotent expansions of LAAs

Let **C** be a λ-algebra. Recall that, for all combinatory polynomials $t, u \in P_I(C)$, we have that $t \equiv_{\mathbf{C}} u$ holds if and only if the equation $t = u$ can be derived from the axioms of combinatory algebra and the equational diagram of **C**. Krivine [26] (see also [42, Chapter 2]) has found an interesting algebraic proof of Lemma 37, that, for all λ-algebras **C** and all combinatory polynomials t and u, $t \equiv_{\mathbf{C}} u$ if and only if $\lambda^* x(t) \equiv_{\mathbf{C}} \lambda^* x(u)$. In this section we briefly review Krivine's construction and extend it to lambda abstraction algebras. We show that every LAA_I **A** is a retract of each of its *idempotent expansions*. An idempotent expansion of **A** is an algebra in the similarity type of LAA_Is having the range of a *good idempotent* of **A** as its universe and suitable term operations of **A** as operations. Among the idempotent expansions of **A**, there is a minimal one called the least idempotent expansion of **A**. We show that the least idempotent expansion of **A** is an LAA_I, which verifies the same identities of **A**. In the last part of the section we show that the lattice of congruences of every LAA_I **A** is isomorphic to the lattice of congruences of the algebra induced by **A** on the range of every good idempotent.

By the combinatory completeness lemma (Proposition 31) we have that, for every combinatory polynomial $t(x)$ in one λ-variable x, there exists an element $c = \lambda^* x(t)^{\mathbf{C}}$ in C such that

$$t(x) \equiv_{\mathbf{C}} \bar{c}x.$$

Moreover, if $t(x) \equiv_{\mathbf{C}} \bar{c}x$ and $u(x) \equiv_{\mathbf{C}} \bar{d}x$ then

$$t(x) \cdot u(x) \equiv_{\mathbf{C}} \overline{\mathbf{s}cd}x.$$

Since the constant polynomial functions having values $\mathbf{k}$ and $\mathbf{s}$ can be written as

$$\overline{\mathbf{kk}}x; \quad \overline{\mathbf{ks}}x,$$

then it is quite natural to introduce an algebra in the similarity type of combinatory algebras

$$\mathbf{C}' = \langle C, \bullet, \mathbf{kk}, \mathbf{ks} \rangle, \tag{5.1}$$

where $c \bullet d = \mathbf{s}cd$ for all $c, d \in C$. In [26] it was shown that the set $\mathbf{1}C = \{c : \mathbf{1}^{\mathbf{C}}c = c\}$, where $\mathbf{1} = \mathbf{s}(\mathbf{ki})$, is a subuniverse of $\mathbf{C}'$, and that the corresponding subalgebra $\mathbf{1C}$ is the free extension of $\mathbf{C}$ by an indeterminate in the variety of combinatory algebras. More precisely, the map $\iota : \mathbf{C} \to \mathbf{1C}$, defined by $\iota(a) = \mathbf{k}a$ for all $a \in C$, is an embedding of $\mathbf{C}$ into $\mathbf{1C}$, and for every homomorphism $f : \mathbf{C} \to \mathbf{B}$ of combinatory algebras and every $b \in B$, there exists a unique homomorphism $g : \mathbf{1C} \to \mathbf{B}$ such that $g \circ \iota = f$ and $g(\mathbf{i}) = b$. The map g is defined by $g(a) = f(a) \cdot^{\mathbf{B}} b$ for all $a \in \mathbf{1}C$. The isomorphism from $\mathbf{1C}$ onto the free extension $\mathbf{C}[x]$ of $\mathbf{C}$ in the variety of combinatory algebras can be defined by extending the embedding f of $\mathbf{C}$ into $\mathbf{C}[x]$

$$f(c) = [\overline{\mathbf{k}c}x]_{\equiv_{\mathbf{C}}}$$

to the isomorphism $g : \mathbf{1C} \to \mathbf{C}[x]$ such that

$$g(c) = f(c) \cdot^{\mathbf{C}[x]} [x]_{\equiv_{\mathbf{C}}} = [\overline{\mathbf{k}c}x]_{\equiv_{\mathbf{C}}} \cdot^{\mathbf{C}[x]} [\bar{\mathbf{i}}x]_{\equiv_{\mathbf{C}}} = [\overline{\mathbf{s}(\mathbf{k}c)\mathbf{i}}x]_{\equiv_{\mathbf{C}}} = [\overline{\mathbf{1}c}x]_{\equiv_{\mathbf{C}}} = [\bar{c}x]_{\equiv_{\mathbf{C}}}.$$

The inverse homomorphism maps $[\bar{c}x]_{\equiv_{\mathbf{C}}}$ into $\mathbf{1}c$. As previously said, Krivine utilizes the above construction for giving an algebraic proof of Lemma 37.

In the remaining part of this section we assume that the dimension set I is infinite. As usual, in the similarity type of LAA_Is let $\mathbf{1} = \lambda xy.xy$, $\mathbf{s} = \lambda xyz.xz(yz)$, $\mathbf{k} = \lambda xy.x$ and $\mathbf{i} = \lambda x.x$.

The definition of the operation $\lambda x'$ in Definition 48 below was suggested by a referee.

Definition 48. Let $\mathbf{A}$ be an LAA_I. Define an algebra $\mathbf{A}'$ in the same similarity type of LAA_Is

$$\mathbf{A}' = \langle A, \cdot', \lambda x', x' \rangle_{x \in I}$$

as follows, for all $a, b \in A$:
(i) $a \cdot' b = \mathbf{s}^{\mathbf{A}}ab$;
(ii) $\lambda x'.a = \mathbf{s}^{\mathbf{A}}(\mathbf{k}^{\mathbf{A}}(\mathbf{s}^{\mathbf{A}}(\lambda x^{\mathbf{A}}.a)))\mathbf{k}^{\mathbf{A}}$;
(iii) $x' = \mathbf{k}^{\mathbf{A}}x^{\mathbf{A}}$.
The algebra $\mathbf{A}'$ is called the *top idempotent expansion* of $\mathbf{A}$.

Note. ab denotes application in $\mathbf{A}$, i.e., $ab = a \cdot^{\mathbf{A}} b$; $a \cdot' b$ will denote application in $\mathbf{A}'$.

If a fresh λ-variable z with respect to a and x is available, then we can also write $\lambda x'.a$ as follows:

$$\lambda x'.a = \lambda z x^{\mathbf{A}}.az^{\mathbf{A}}, \quad \text{if } z \notin \Delta a \cup \{x\}. \tag{5.2}$$

A term operation on an LAA_I $\mathbf{A}$ is a function $f: A^n \to A$ for which there exists a λ-context $t(\xi_1, \ldots, \xi_n)$ such that

$$f(a_1, \ldots, a_n) = t^{\mathbf{A}}(a_1, \ldots, a_n), \quad \text{for all } a_1, \ldots, a_n \in A.$$

The set of n-ary term operations is denoted by $\mathrm{Clo}_n\, \mathbf{A}$.

Definition 49. Let $\mathbf{A}$ be an LAA_I. The set of $e \in \mathrm{Clo}_1 \mathbf{A}$ of all unary term operations such that $e = e^2 (= e \circ e)$ will be denoted by $E(\mathbf{A})$. Each element in $E(\mathbf{A})$ is called an *idempotent*. An idempotent e is *good* if the following identities hold for every $a, b \in A$

$$e\mathbf{k} = \mathbf{k}; \quad e(\mathbf{k}a) = \mathbf{k}a; \quad e\mathbf{s} = \mathbf{s}; \quad e(\mathbf{s}a) = \mathbf{s}a; \quad e(\mathbf{s}ab) = \mathbf{s}ab.$$

For every idempotent e, eA will denote the range of the map e, i.e.,

$$eA = \{a : e(a) = a\} = \{e(a) : a \in A\}.$$

The λ-contexts $\mathbf{i}\xi$ and $\mathbf{1}\xi$ determine good idempotents on $\mathbf{A}$, while the idempotent defined by the λ-context $(\lambda x.\xi)z$ $(x \neq z)$ is not good.

The proof of the following proposition is an easy consequence of the definition of good idempotent.

Proposition 50. *Let* $\mathbf{A}'$ *be the top idempotent expansion of an* LAA_I $\mathbf{A}$. *If* e *is a good idempotent then the set* eA *is a subuniverse of* $\mathbf{A}'$.

We denote by $e\mathbf{A}$ the subalgebra of $\mathbf{A}'$ associated with the subuniverse eA. We call each $e\mathbf{A}$ an *idempotent expansion* of $\mathbf{A}$.

Denote by $\mathbf{1A}$ and $\mathbf{iA}$ the idempotent expansions of $\mathbf{A}$ determined, respectively, by the λ-contexts $\mathbf{1}\xi$ and $\mathbf{i}\xi$.

Proposition 51. *The idempotent expansions of* $\mathbf{A}$ *partially ordered by the relation* $\subseteq$ *have* $\mathbf{1A}$ *as a minimum and* $\mathbf{iA}$ *as a maximum.*

Proof. $\mathbf{iA}$ is just the top idempotent expansion $\mathbf{A}'$ of $\mathbf{A}$. Recall that $\mathbf{1} = \mathbf{s}(\mathbf{ki})$. If e is a good idempotent and $\mathbf{1}a = a$ then $e(a) = e(\mathbf{s}(\mathbf{ki})a) = \mathbf{s}(\mathbf{ki})a = a$. $\square$

The algebra $\mathbf{1A}$ is called the *least idempotent expansion* of $\mathbf{A}$.

Proposition 52. *The least idempotent expansion* $\mathbf{1A}$ *of* $\mathbf{A}$ *is both a subalgebra and a homomorphic image of the top idempotent expansion* $\mathbf{A}'$.

Proof. We know from Proposition 50 that $\mathbf{1A}$ is a subalgebra of $\mathbf{A}'$. We prove the remaining part of the proposition. Define a map $f: A \to \mathbf{1}A$ as follows: $f(a) = \mathbf{1}a$.

Recall that in every λ-algebra, the equation $\mathbf{1}(\mathbf{s}ab) = \mathbf{s}ab = \mathbf{s}(\mathbf{1}a)(\mathbf{1}b)$ holds. Then we have

$$f(a \cdot' b) = \mathbf{1}(\mathbf{s}ab) = \mathbf{s}ab = \mathbf{s}(\mathbf{1}a)(\mathbf{1}b) = f(a) \cdot' f(b)$$

and

$$f(x') = \mathbf{1}(\mathbf{k}x) = \mathbf{k}x = x'.$$

By definition of $\lambda x'$ we have that $f(\lambda x'.a) = \lambda x'.f(a)$ if, and only if, $\mathbf{A}$ satisfies the identity

$$\mathbf{1}[\mathbf{s}(\mathbf{k}(\mathbf{s}(\lambda x.\xi)))\mathbf{k}] = \mathbf{s}(\mathbf{k}(\mathbf{s}(\lambda x.\mathbf{1}\xi)))\mathbf{k}. \tag{5.3}$$

Since the variety LAA_I is generated by the term algebra of the minimal lambda theory $\lambda\beta$ (Theorem 14), it is sufficient to prove (5.3) when ξ ranges over the set F_I of λ-terms. Let $t \in F_I$ and let z be a fresh λ-variable with respect to t and x. Then we have

$$\begin{aligned}
\mathbf{1}[\mathbf{s}(\mathbf{k}(\mathbf{s}(\lambda x.t)))\mathbf{k}] &= \mathbf{1}(\lambda zx.tz) \quad [(5.2)]\\
&= \lambda zx.tz\\
&= \lambda zx.\mathbf{1}tz \quad [\mathbf{1}tz = tz]\\
&= \mathbf{s}(\mathbf{k}(\mathbf{s}(\lambda x.\mathbf{1}t)))\mathbf{k} \quad [(5.2)].
\end{aligned}$$

Proposition 53. $\mathbf{A}$ *is a retract of every idempotent expansion* $e\mathbf{A}$ *of* $\mathbf{A}$ *via the pair* $\langle \imath : \mathbf{A} \to e\mathbf{A}, \gamma : e\mathbf{A} \to \mathbf{A}\rangle$ *of homomorphisms defined by*

$$\imath(a) = \mathbf{k}^{\mathbf{A}}a \quad \textit{for all } a \in A$$

and

$$\gamma(a) = a\mathbf{i}^{\mathbf{A}} \quad \textit{for all } a \in eA.$$

Proof. We prove the result for the top idempotent expansion $\mathbf{A}'$. First, we show that $\imath$ is a homomorphism. Let $a, b \in A$. Recall that the combinatory reduct of every LAA is a λ-algebra and then it satisfies the following identity for all $a, b \in A$.

$$\mathbf{s}(\mathbf{k}a)(\mathbf{k}b) = \mathbf{k}(ab). \tag{5.4}$$

We have

$$\begin{aligned}
\imath(ab) &= \mathbf{k}(ab)\\
&= \mathbf{s}(\mathbf{k}a)(\mathbf{k}b) \quad [\text{by } (5.4)]\\
&= \imath(a) \cdot' \imath(b)
\end{aligned}$$

and

$$\begin{aligned}\imath(x) &= \mathbf{k}x\\ &= x'.\end{aligned}$$

By definition of $\lambda x'$ we have that $\imath(\lambda x.a) = \lambda x'.\imath(a)$ if the LAA_I $\mathbf{A}$ satisfies the identity

$$\mathbf{s}(\mathbf{k}(\mathbf{s}(\lambda x.\mathbf{k}\xi)))\mathbf{k} = \mathbf{k}(\lambda x.\xi). \tag{5.5}$$

Since the variety LAA_I is generated by the term algebra of the minimal lambda theory $\lambda\beta$ (Theorem 14), it is sufficient to prove (5.5) when ξ ranges over the set F_I of λ-terms. Let $t \in F_I$ and let z be a fresh λ-variable with respect to t and x.

$$\begin{aligned}\mathbf{s}(\mathbf{k}(\mathbf{s}(\lambda x.\mathbf{k}t)))\mathbf{k} &= \lambda zx.\mathbf{k}tz \quad [(5.2)]\\ &= \lambda zx.t \quad [\mathbf{k}tz = t]\\ &= \mathbf{k}(\lambda x.t) \quad [z \text{ fresh}].\end{aligned}$$

We now show that γ is a homomorphism:

$$\begin{aligned}\gamma(a \cdot' b) &= \mathbf{s}ab\mathbf{i}\\ &= a\mathbf{i}(b\mathbf{i})\\ &= \gamma(a) \cdot^{\mathbf{A}} \gamma(b),\end{aligned}$$

$$\begin{aligned}\gamma(x') &= \mathbf{k}x\mathbf{i}\\ &= x.\end{aligned}$$

Since $\gamma(\lambda x'.a) = \mathbf{s}(\mathbf{k}(\mathbf{s}(\lambda x.a)))\mathbf{k}\mathbf{i}$ and $\lambda x.\gamma(a) = \lambda x.a\mathbf{i}$, the conclusion can be obtained if we show that the LAA_I $\mathbf{A}$ satisfies the identity

$$\mathbf{s}(\mathbf{k}(\mathbf{s}(\lambda x.\xi)))\mathbf{k}\mathbf{i} = \lambda x.\xi\mathbf{i}. \tag{5.6}$$

The verification of this identity is similar to that of (5.5).

Finally, if $a \in A$ then $\gamma(\imath(a)) = \gamma(\mathbf{k}a) = \mathbf{k}a\mathbf{i} = a$. □

We now characterize the identities between λ-contexts satisfied by the class of the idempotent expansions of LAA_Is.

The author is indebted to a referee for most of the results included between Lemma 54 and Theorem 62.

If $t = t(\xi_1, \ldots, \xi_n)$ is a λ-context, we denote by $t' = t'(\xi_1, \ldots, \xi_n)$ a new λ-context defined by induction as follows:

$$(\xi_i)' = \xi_i,$$

$$x' = \mathbf{k}x,$$

$$(tu)' = \mathbf{s}t'u',$$

$$(\lambda x.u)' = \mathbf{s}(\mathbf{k}(\mathbf{s}(\lambda x.u')))\mathbf{k}.$$

Lemma 54. *Let* $\mathbf{A}$ *be an* LAA_I. *Then we have*

$$\mathbf{A}' \models t(\bar{\xi}) = u(\bar{\xi}) \text{ iff } \mathbf{A} \models t'(\bar{\xi}) = u'(\bar{\xi}).$$

Proof. By definition of t' we have, for all $a_1, \ldots, a_n \in A$,

$$t^{\mathbf{A}'}(a_1, \ldots, a_n) = (t')^{\mathbf{A}}(a_1, \ldots, a_n). \qquad \square \tag{5.7}$$

A λ-context $t = t(\bar{\xi})$ is a *projection* if it is of the form $t = \xi_i$, for some i. Any λ-context that is not a projection is of one of the forms $t \cdot u$, $\lambda x.t$, or x, where x is a λ-variable.

Definition 55. An identity $t(\bar{\xi}) = u(\bar{\xi})$ is called *normal* if it is either the identity $\xi_i = \xi_i$ or both t and u are not projections.

Lemma 56. *Let* $\mathbf{A}$ *be an* LAA_I *and* t *be a* λ*-context. If* t *is not a projection, then we have*

$$\mathbf{A} \models t' = \mathbf{1}t'. \tag{5.8}$$

Proof. For all $a, b \in A$ and for all $x \in I$ we have

$$a \cdot' b = \mathbf{s}ab = \mathbf{1}(a \cdot' b),$$

$$\lambda x'.a = \mathbf{s}(\mathbf{k}(\mathbf{s}(\lambda x.a)))\mathbf{k} = \mathbf{1}(\lambda x'.a),$$

$$x' = \mathbf{k}x = \mathbf{1}(x'). \qquad \square$$

A lambda theory T is *extensional* if it is closed under the following rule of extensionality [3, Definition 2.1.27]

$$(\mathbf{ext}) \quad tx = sx \in T \quad \text{with } x \text{ not free in } ts \Rightarrow t = s \in T.$$

The rule of extensionality is equivalent to (η)-conversion: $\lambda x.tx = t$ with x not free in t.

The following is the algebraic version of (η)-conversion.

Definition 57. We say that an LAA_I is *extensional* if it satisfies the following identity:

$$(\eta) \ \lambda x.(\lambda x.\xi)yx = (\lambda x.\xi)y, \quad x \neq y.$$

Proposition 58. *An* LAA_I $\mathbf{A}$ *is extensional if, and only if, it satisfies one of the following equivalent conditions*:

(i) $\mathbf{i} = \mathbf{1}$.

(ii) *For every* λ*-variable* y, $y = \lambda x.a$ $(y \neq x)$ *for some* $a \in A$.

(iii) *The top idempotent expansion* $\mathbf{A}'$ *of* $\mathbf{A}$ *satisfies at least one identity that is not normal.*

Proof. $(\eta) \Rightarrow$ (i): $\lambda y.xy = x$ by (η) and by $(\lambda x.y)x = y$.

(i) $\Rightarrow (\eta)$: Let $a = (\lambda x.b)y$ $(x \neq y)$. Then, $\lambda x.ax = (\lambda yx.yx)a = \mathbf{1}a = \mathbf{i}a = a$.

(i) $\Rightarrow$ (ii): $y = \mathbf{i}y = \mathbf{1}y = (\lambda yx.yx)y = \lambda x.yx$.

(ii) $\Rightarrow$ (i): If $y = \lambda x.a$ then $yx = (\lambda x.a)x = a$ by (β_3). Then $\lambda x.yx = \lambda x.a = y$ so that $\lambda yx.yx = \lambda y.y$.

(iii) $\Rightarrow$ (i): Let $t(\bar{\xi}) = \xi_1$ be any non-normal identity that holds in $\mathbf{A}'$. From Lemma 54 it follows that the identity $t'(\bar{\xi}) = \xi_1$ is valid for $\mathbf{A}$, from which $\mathbf{A} \models t'(x, \xi_2, \ldots, \xi_n) = x$, where x is a λ-variable. Since t is not a projection, Lemma 56 implies

$$\mathbf{A} \models x = t'(x, \xi_2, \ldots, \xi_n) = \mathbf{1}t'(x, \xi_2, \ldots, \xi_n) = \mathbf{1}x,$$

which is equivalent to $\lambda x.x = \mathbf{1}$.

(i) $\Rightarrow$ (iii): It is sufficient to prove that $\mathbf{A}' \models (\lambda x'.x') \cdot' \xi = \xi$. Let $z \neq x$ be a λ-variable. Then

$$\begin{aligned}
(\lambda x'.x') \cdot' \xi &= \mathbf{s}(\lambda zx.\mathbf{k}xz)\xi \quad [(5.2)] \\
&= \mathbf{s}(\lambda zx.x)\xi \quad [\mathbf{k}xz = x] \\
&= \mathbf{s}(\mathbf{ki})\xi \quad [\lambda x.x = \mathbf{i}] \\
&= \mathbf{1}\xi \quad [\text{def } \mathbf{1}] \\
&= \xi \quad [\text{by assumption}]. \qquad \square
\end{aligned}$$

Notice that, for all $a, b, c \in A$,

$$(a \cdot' b)c = ac(bc),$$

$$(\lambda x'.a)c = \lambda x.ac \quad \text{if } x \notin \Delta c,$$

$$x'c = x.$$

Hence, by induction, for all λ-contexts $t(\bar{\xi})$ over $\bar{x}$, and all $a_1, \ldots, a_n, c \in A$ such that $\bar{x} \notin \Delta c$,

$$(t')^{\mathbf{A}}(a_1, \ldots, a_n)c = t^{\mathbf{A}}(a_1 c, \ldots, a_n c). \tag{5.9}$$

Lemma 59. *Let* $\mathbf{A}$ *be an* LAA_I *and let* $t(\bar{\xi}) = u(\bar{\xi})$ *be any identity between* λ*-contexts over* $\bar{x}$ *that holds in* $\mathbf{A}$. *Then* $\mathbf{A}$ *satisfies the identity* $\mathbf{1}t'(\bar{\xi}) = \mathbf{1}u'(\bar{\xi})$.

Proof. We recall that an element a of $\mathbf{A}$ is finite dimensional if the dimension set of a is finite. By using Proposition 5 it is simple to prove that the set $\mathrm{Fi}\,A = \{a \in A : |\Delta a| < \omega\}$ is a subuniverse of $\mathbf{A}$. In the following $\mathrm{Fi}\,\mathbf{A}$ denotes the subalgebra of all finite dimensional elements of $\mathbf{A}$.

We first show that the identity $\mathbf{1}t'(\bar{\xi}) = \mathbf{1}u'(\bar{\xi})$ holds in Fi $\mathbf{A}$: for any $\bar{b} = b_1, \dots, b_n \in$ Fi A, one can choose $z \notin \bar{x} \cup \Delta\bar{b}$ to get

$$
\begin{aligned}
\mathbf{1}t'(\bar{b}) &= \lambda z.t'(\bar{b})z \quad [\text{because } z \notin \bar{x} \cup \Delta\bar{b}] \\
&= \lambda z.t(b_1 z, \dots, b_n z) \quad [\text{by } (5.9)] \\
&= \lambda z.u(b_1 z, \dots, b_n z) \quad [\text{by assumption}] \\
&= \lambda z.u'(\bar{b})z \quad [\text{by } (5.9)] \\
&= \mathbf{1}u'(\bar{b}) \quad [\text{because } z \notin \bar{x} \cup \Delta\bar{b}].
\end{aligned}
$$

Thus, $\mathbf{1}t'(\bar{\xi}) = \mathbf{1}u'(\bar{\xi})$ holds in Fi $\mathbf{A}$, and then in the minimal subalgebra of $\mathbf{A}$, which is the term algebra of a suitable lambda theory T over F_I. By Theorem 16 $\mathbf{A}$ is generic in the variety generated by $\mathbf{F}_I^T$, so that $\mathbf{1}t'(\bar{\xi}) = \mathbf{1}u'(\bar{\xi})$ holds in $\mathbf{A}$. □

Theorem 60. *Let* $\mathbf{A}$ *be an* LAA_I:
(i) *If* $\mathbf{A}$ *is not extensional, then* $\mathbf{A}'$ *and* $\mathbf{A}$ *satisfy the same normal identities between* λ*-contexts.*
(ii) *If* $\mathbf{A}$ *is extensional, then* $\mathbf{A}'$ *and* $\mathbf{A}$ *satisfy the same identities between* λ*-contexts.*

Proof. By Proposition 53 $\mathbf{A}$ is a homomorphic image of $\mathbf{A}'$, so that every identity valid for $\mathbf{A}'$ is also valid for $\mathbf{A}$.

Let $t(\bar{\xi}) = u(\bar{\xi})$ be any identity between λ-contexts that holds in $\mathbf{A}$. By Lemma 59 we have that $\mathbf{A} \models \mathbf{1}t'(\bar{\xi}) = \mathbf{1}t'(\bar{\xi})$. By applying Lemma 56 in the hypothesis that $t(\bar{\xi}) = u(\bar{\xi})$ is normal or the identity $\mathbf{1} = \mathbf{i}$ in the hypothesis that $\mathbf{A}$ is extensional we obtain $\mathbf{A} \models t'(\bar{\xi}) = u'(\bar{\xi})$. The conclusion $\mathbf{A}' \models t(\bar{\xi}) = u(\bar{\xi})$ follows from Lemma 54. □

Corollary 61. *An identity* $t(\bar{\xi}) = u(\bar{\xi})$ *is valid in the class of idempotent expansions of* LAA_I*s if, and only if, it is valid in* LAA_I *and it is normal.*

Proof. Every idempotent expansion of an LAA_I $\mathbf{A}$ is a subalgebra of the top idempotent expansion $\mathbf{A}'$ of $\mathbf{A}$. □

Thus the class of idempotent expansions of LAA_Is satisfies the axioms (β_2), (β_4), (β_5), (β_6), (α) and the following weak versions of (β_1) and (β_3), for all $x, y \in I$,
$(w\beta_1)$ $(\lambda x.x)(\lambda y.\xi) = (\lambda y.\xi)$; $(\lambda x.x)y = y$; $(\lambda x.x)\xi = (\lambda y.y)\xi$.
$(w\beta_3)$ $(\lambda x.\lambda y.\xi)x = \lambda y.\xi$; $(\lambda x.y)x = y$; $(\lambda x.\xi)x = (\lambda y.\xi)y$.

Theorem 62. *Let* $\mathbf{A}$ *be an* LAA_I. *Then* $\mathbf{1A}$ *and* $\mathbf{A}$ *satisfy the same identities between* λ*-contexts*; *thus,* $\mathbf{1A}$ *is an* LAA_I.

Proof. By Proposition 53 $\mathbf{A}$ is a homomorphic image of $\mathbf{1A}$, so that every identity satisfied by $\mathbf{1A}$ is also satisfied by $\mathbf{A}$. For the converse, let $t(\bar{\xi}) = \xi_1$ be any nonnormal identity satisfied by $\mathbf{A}$. By Lemma 54 and by Lemma 56 $\mathbf{A} \models t'(\bar{\xi}) = \mathbf{1}\xi_1$. Let

$a_1,\dots,a_n \in \mathbf{1}A$.

$$\begin{aligned} t^{\mathbf{1A}}(a_1,\dots,a_n) &= t^{\mathbf{A}'}(a_1,\dots,a_n) \quad [\mathbf{1A} \subseteq \mathbf{A}'] \\ &= (t')^{\mathbf{A}}(a_1,\dots,a_n) \quad [(5.7)] \\ &= \mathbf{1}a_1 \\ &= a_1 \quad [a_1 \in \mathbf{1}A]. \end{aligned}$$

Thus $\mathbf{1A}$ satisfies every non-normal identity satisfied by $\mathbf{A}$. The conclusion follows from Theorem 60(i). □

5.1. *A characterization of the lattice of congruences of* LAA*s*

A polynomial operation on $\mathbf{A}$ is a function $f: A^n \to A$ for which there exist a λ-context $t(\xi_1,\dots,\xi_n,\mu_1,\dots,\mu_k)$ and elements $b_1,\dots,b_k \in A$ such that

$$f(a_1,\dots,a_n) = t^{\mathbf{A}}(a_1,\dots,a_n,b_1,\dots,b_k) \quad \text{for all } a_1,\dots,a_n \in A.$$

The set of (n-ary) polynomial operations is denoted by $\mathrm{Pol}\,\mathbf{A}$ ($\mathrm{Pol}_n\,\mathbf{A}$).

Suppose that U is a nonempty subset of A. Then we define:

(i) $(\mathrm{Pol}\,\mathbf{A})|_U$ is the set of all $h|_U$ such that $h \in \mathrm{Pol}_n\,\mathbf{A}$ for some natural n, and $h(U^n) \subseteq U$;

(ii) $\mathbf{A}|_U = \langle U, (\mathrm{Pol}\,\mathbf{A})|_U\rangle$ is called *the algebra induced on U* by $\mathbf{A}$.

For every good idempotent e, the algebra $e\mathbf{A}$ is a reduct of the algebra $\mathbf{A}|_{eA}$ induced on eA by $\mathbf{A}$.

We now show that, for every LAA_I $\mathbf{A}$ and every good idempotent e, there exists a complete lattice isomorphism between the lattice $\mathbf{Con}\,\mathbf{A}$ (of congruences on $\mathbf{A}$) and the lattice $\mathbf{Con}\,\mathbf{A}|_{eA}$ of congruences on $\mathbf{A}|_{eA}$.

Theorem 63. *Supppose that $\mathbf{A}$ is an LAA_I and $e \in E(\mathbf{A})$ is a good idempotent. Then we have that the mapping $|_{eA}$ defined by*

$$\vartheta \in \mathbf{Con}\,\mathbf{A} \mapsto \vartheta|_{eA} \in \mathbf{Con}\,\mathbf{A}|_{eA}$$

is a complete lattice isomorphism of $\mathbf{Con}\,\mathbf{A}$ onto $\mathbf{Con}\,\mathbf{A}|_{eA}$. The inverse mapping is defined for all congruences $\gamma \in \mathbf{Con}\,\mathbf{A}|_{eA}$ as follows:

$$\gamma \mapsto \gamma^{-1} = \{(a,b) \in A^2\colon\ (e(f(a)), e(f(b))) \in \gamma \textit{ for all } f \in Pol_1\mathbf{A}\}.$$

Proof. Pálfy and Pudlák [28, Lemma 4.22] have shown that for any algebra $\mathbf{A}$ and idempotent e the mapping $|_{eA}$ is a complete lattice homomorphism of $\mathbf{Con}\,\mathbf{A}$ onto $\mathbf{Con}\,\mathbf{A}|_{eA}$. Then it is sufficient to prove that $|_{eA}$ is one-to-one. Let θ and ϑ be congruences on $\mathbf{A}$ such that $\theta|_{eA} = \vartheta|_{eA}$. We are going to show that, for all $a,b \in A$, we have

$$a\ \theta\ b \quad \text{iff } \mathbf{k}a\ \theta|_{eA}\ \mathbf{k}b.$$

If $\mathbf{k}a\ \theta|_{eA}\ \mathbf{k}b$ then $\mathbf{k}a\ \theta\ \mathbf{k}b$, so that by the congruence properties of θ we have $\mathbf{k}ac\ \theta\ \mathbf{k}bc$, and then $a\ \theta\ b$. In conclusion we obtain

$$a\ \theta\ b \Leftrightarrow \mathbf{k}a\ \theta\ \mathbf{k}b \Leftrightarrow \mathbf{k}a\ \theta|_{eA}\ \mathbf{k}b \Leftrightarrow \mathbf{k}a\ \vartheta|_{eA}\ \mathbf{k}b \Leftrightarrow \mathbf{k}a\ \vartheta\ \mathbf{k}b \Leftrightarrow a\ \vartheta\ b. \qquad \square$$

The above theorem implies that we have a meet preserving embedding $|_{eA}$ from $\mathbf{Con\,A}$ into $\mathbf{Con}\,e\mathbf{A}$

$$\vartheta \in \mathbf{Con\,A} \mapsto \vartheta|_{eA} \in \mathbf{Con}\,e\mathbf{A}.$$

If ϑ is a congruence on $e\mathbf{A}$ we denote by $\vartheta^\bullet$ the congruence on $\mathbf{A}$ defined by

$$a\ \vartheta^\bullet\ b \quad \text{iff } \mathbf{k}a\ \vartheta\ \mathbf{k}b \quad \text{for all } a,b \in A.$$

We have that $(\vartheta|_{eA})^\bullet = \vartheta$ for all $\vartheta \in \mathbf{Con\,A}$, and $(\vartheta^\bullet)|_{eA} = \mathbf{k}^{-1}\vartheta$ for all $\vartheta \in \mathbf{Con}\,e\mathbf{A}$, where $a\ \mathbf{k}^{-1}\vartheta\ b$ iff $\mathbf{k}a\ \vartheta\ \mathbf{k}b$. If $\vartheta \in \mathbf{Con\,A}|_{eA}$ we have that $(\vartheta^\bullet)|_{eA} = \vartheta$ and hence $\vartheta^{-1} = \vartheta^\bullet$, where ϑ^{-1} is the congruence defined in Theorem 63.

As a matter of notation, if $\mathbf{A}$ is an algebra, $0_\mathbf{A}$ and $1_\mathbf{A}$ are respectively the least element and the greatest element of $\mathbf{Con\,A}$. If $\vartheta \subseteq \theta$ are congruences, then $[\vartheta, \theta] = \{\eta : \vartheta \subseteq \eta \subseteq \theta\}$ is a complete sublattice of $\mathbf{Con\,A}$.

We have shown in Proposition 53 that the mapping r, defined by $r(a) = a\mathbf{i}$, is a homomorphism from $e\mathbf{A}$ onto $\mathbf{A}$. So, $\mathbf{A}$ is a homomorphic image of $e\mathbf{A}$. Define for all $a,b \in e(A)$

$$a\ \theta_r\ b \quad \text{iff } r(a) = a\mathbf{i} = b\mathbf{i} = r(b).$$

Then we have

Proposition 64. *The mapping* $^\bullet$ *defines a lattice isomorphism*

$$^\bullet\colon\ [\theta_r, 1_{e\mathbf{A}}] \cong \mathbf{Con\,A}.$$

Proof. First, we show that $\theta_r^\bullet = 0_\mathbf{A}$:

$$a\ \theta_r^\bullet\ b \Leftrightarrow \mathbf{k}a\ \theta_r\ \mathbf{k}b \Leftrightarrow \mathbf{k}a\mathbf{i} = \mathbf{k}b\mathbf{i} \Leftrightarrow a = b.$$

Let $\theta_r \subseteq \vartheta$. Define the congruence $\equiv$ on $\mathbf{A}$ as follows:

$$r(a) \equiv r(b) \Leftrightarrow a\ \vartheta\ b.$$

Since $e\mathbf{A}/\theta_r$ is isomorphic to the subalgebra $\imath(\mathbf{A})$ of $e\mathbf{A}$ (for every element $a \in e(A)$ we have $a\ \theta_r\ \mathbf{k}(a\mathbf{i})$), then we have for all $a,b \in A$: $a \equiv b$ iff $a'\ \vartheta\ b'$ for some $a', b' \in e(A)$ such that $a'\mathbf{i} = a$ and $b'\mathbf{i} = b$, iff $\mathbf{k}(a'\mathbf{i})\ \vartheta\ \mathbf{k}(b'\mathbf{i})$ iff $\mathbf{k}a\ \vartheta\ \mathbf{k}b$. It follows that $\equiv\, = \vartheta^\bullet$. $\square$

We conclude the section by showing that, for every LAA_I $\mathbf{A}$, the algebras $\mathbf{A}^n$ are subreducts of the algebra $\mathbf{A}_{|A}$.

There exists a combinator $\mathbf{c} = \lambda xyz^*(zxy)$ such that the identity

$$\mathbf{c}xyz = zxy$$

holds in all combinatory algebras. Let $\mathbf{A}$ be an LAA_I and let $\mathbf{T} = \lambda xy.x$, $\mathbf{F} = \lambda xy.y$. Recall the combinatory reduct of $\mathbf{A}$ is a combinatory algebra.

Define the mapping $p_2 : A \to A$ as follows:

$$p_2(a) = \mathbf{c}(a\mathbf{T})(a\mathbf{F}) \quad \text{for all } a \in A.$$

It is obvious that p_2 is a unary polynomial on $\mathbf{A}$ and that p_2 is idempotent:

$$\begin{aligned} p_2 p_2(a) &= \mathbf{c}[p_2(a)\mathbf{T}][p_2(a)\mathbf{F}] \\ &= \mathbf{c}[\mathbf{c}(a\mathbf{T})(a\mathbf{F})\mathbf{T}][\mathbf{c}(a\mathbf{T})(a\mathbf{F})\mathbf{F}] \\ &= \mathbf{c}[\mathbf{T}(a\mathbf{T})(a\mathbf{F})][\mathbf{F}(a\mathbf{T})(a\mathbf{F})] \\ &= \mathbf{c}(a\mathbf{T})(a\mathbf{F}) \\ &= p_2(a). \end{aligned}$$

Define an algebra $p_2\mathbf{A}$ of domain $p_2(A)$ as follows for all $a, b \in p_2(A)$:

(i) $\lambda x^{p_2\mathbf{A}}.a = \mathbf{c}(\lambda x^{\mathbf{A}}.a\mathbf{T})(\lambda x^{\mathbf{A}}.a\mathbf{F})$;
(ii) $a \cdot^{p_2\mathbf{A}} b = \mathbf{c}(a\mathbf{T}(a\mathbf{F}))(b\mathbf{T}(b\mathbf{F}))$;
(iii) $x^{p_2\mathbf{A}} = \mathbf{c}x^{\mathbf{A}}x^{\mathbf{A}}$.

Proposition 65. *$p_2\mathbf{A}$ is an LAA_I isomorphic to $\mathbf{A} \times \mathbf{A}$.*

Proof. We define an isomorphism $f : A \times A \to p_2(A)$ as follows for all $a, b \in A$:

$$f(a, b) = \mathbf{c}ab.$$

The map f is one-to-one because $a = \mathbf{c}ab\mathbf{T}$ and $b = \mathbf{c}ab\mathbf{F}$. Map f is a homomorphism:

$$\begin{aligned} f(x^{\mathbf{A}}, x^{\mathbf{A}}) &= \mathbf{c}x^{\mathbf{A}}x^{\mathbf{A}} \\ &= x^{p_2\mathbf{A}}, \end{aligned}$$

$$\begin{aligned} f(\lambda x^{\mathbf{A}}.a, \lambda x^{\mathbf{A}}.b) &= \mathbf{c}(\lambda x^{\mathbf{A}}.a)(\lambda x^{\mathbf{A}}.b) \\ &= \mathbf{c}(\lambda x^{\mathbf{A}}.\mathbf{c}ab\mathbf{T})(\lambda x^{\mathbf{A}}.\mathbf{c}ab\mathbf{F}) \\ &= \lambda x^{p_2\mathbf{A}}.\mathbf{c}ab \\ &= \lambda x^{p_2\mathbf{A}}.f(a, b), \end{aligned}$$

$$\begin{aligned} f(ab, cd) &= \mathbf{c}(ab)(cd) \\ &= \mathbf{c}[(\mathbf{c}ab\mathbf{T})(\mathbf{c}ab\mathbf{F})][(\mathbf{c}cd\mathbf{T})(\mathbf{c}cd\mathbf{F})] \\ &= \mathbf{c}[f(a,b)\mathbf{T}(f(a,b)\mathbf{F})][f(c,d)\mathbf{T}(f(c,d)\mathbf{F})] \\ &= f(a,b) \cdot^{p_2\mathbf{A}} f(c,d). \end{aligned}$$

Proposition 66. *The mapping* $|_{p_2(A)}$ *is a complete lattice isomorphism of* $\mathbf{Con}\,\mathbf{A}$ *onto* $\mathbf{Con}\,\mathbf{A}|_{p_2(A)}$.

Proof. Pálfy and Pudlák [28, Lemma 4.22] have shown that for any algebra $\mathbf{A}$ and idempotent e the mapping $|_{eA}$ is a complete lattice homomorphism of $\mathbf{Con}\,\mathbf{A}$ onto $\mathbf{Con}\,\mathbf{A}|_{eA}$. Since p_2 is idempotent, it is sufficient to prove that $|_{p_2(A)}$ is one-to-one. Let θ and ϑ be congruences on $\mathbf{A}$ such that $\theta|_{p_2(A)} = \vartheta|_{p_2(A)}$. Then we have

$$a\ \theta\ b \Leftrightarrow \mathbf{c}aa\ \theta\ \mathbf{c}bb \Leftrightarrow \mathbf{c}aa\ \vartheta\ \mathbf{c}bb \Leftrightarrow a\ \vartheta\ b. \qquad \Box$$

The above construction can be generalized to every n, that is, there exists an idempotent unary polynomial p_n and an algebra $p_n\mathbf{A}$ of domain $p_n(A)$ which is a reduct of $\mathbf{A}|_{p_n(A)}$ such that $p_n\mathbf{A}$ is isomorphic to the Cartesian product $\mathbf{A}^n$.

6. Conclusion and related work

The way in which lambda abstraction theory arises from the lambda calculus almost exactly parallels the way cylindric and polyadic algebras [22, 21] are obtained from first-order logic. The axioms of first-order logic are like those of lambda calculus in that the formula variables cannot be substituted without restriction. In both cases the source of the problem is the way substitution for individuals is handled. By dealing with substitution at the level of the object language rather than the metalanguage, i.e., by abstracting it, a pure equational formalization of lambda calculus can be developed giving rise to the theory of LAAs. This abstraction of substitution is a characteristic feature of algebraic logic.

Lambda abstraction theory has been extensively developed by Goldblatt, Pigozzi and the author in a series of papers [17, 18, 32–35, 37, 40], and, as in the theory of cylindric and polyadic algebras, the emphasis is on representation results. The most natural LAAs are *functional* algebras (FLAs) consisting of suitable functions obtained from the combinatory models of the lambda calculus. The completeness theorem for the lambda calculus [29], namely every lambda theory consists of precisely the equations valid in some lambda model (or environment model), can be also obtained as a corollary of the functional representation theorem for locally finite LAAs [32]. The axiomatization of FLAs is the central issue in the algebraic approach to the model theory of lambda calculus. In [35, Theorem 7.7] it was shown that the smallest variety of LAA_Is that includes the functional algebras can be characterized as the class of algebras isomorphic to a certain kind of generalized FLAs called point-relativized functional LAA_Is (RFA_I, for short). The RFA_Is turn out to be (up to isomorphism) exactly the LAA_Is that can be neatly embedded in an LAA_I of infinite higher dimension [35, Theorem 7.4]. In the same paper it was stated the open problem if $\mathbb{I}\mathsf{FLA}_I$ is also a variety and hence coincides with $\mathbb{I}\mathsf{RFA}_I$. The conjecture was proven true by Goldblatt [17] in June 1995; he proved that any RFA_I is isomorphic to an FLA_I. Methods from nonstandard analysis were applied by Goldblatt in [18] to give a new proof of this result. Later in joint work Goldblatt and the author [40] used this result to prove that the variety $\mathbb{I}\mathsf{FLA}_I$ of algebras isomorphic to functional LAA_Is is axiomatized by the finitely many schemes of identities defining LAA_Is.

6.1. Related work

There have been several attempts to reformulate the lambda calculus as a purely algebraic theory within the context of category theory: Obtułowicz and Wieger [31] via the *algebraic theories* of Lawvere; Adachi [2] via *monads*; Curien [11] via *categorical combinators*. There have also been several works that present an algebraic theory of the lambda calculus very close to lambda calculus in spirit. Locally finite functional LAAs are very similar to the functional models of the lambda calculus developed in Krivine [26]. However, Krivine's models do not have an explicit algebraic structure. An abstractly defined class of algebras, called lambda substitution algebras, that is even closer in spirit to LAA's has been introduced by Diskin [13, 14].

The theories of cylindric and polyadic algebras are two early contributions to the algebraization of quantifier logics and have greatly influenced our work. The main references for cylindric algebras are [22, 23]; for polyadic algebras it is [21]; see in particular [20]. We also mention here Nemeti [30]. It contains an extensive survey of the various algebraic versions of quantifier logics.

The importance of abstract substitution, and lambda abstraction, has been recognized for some time among computer scientists because it leads among other things to more natural term rewriting systems, which are useful in the analysis of processes of computations. See for example [1]. In the *transformation algebras* and *substitution algebras* of LeBlanc [27] and Pinter [38] substitution is primitive and abstract quantification is defined in terms of it. A pure theory of abstract substitution has been developed by Feldman [15, 16].

Finally, we mention some recent work of ours connecting a theory of substitution in combination with abstract variable-binding operators. See [36, 39].

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